Topology II
Exam Tuesday October 21, 2025
Exam time 14.00-17.00

In the following problems $\mathbb{R}^n$ is given the Euclidean topology unless otherwise specified. Product spaces $\prod_{i\in I} X_i$ are given the product topology induced from spaces X_i and the quotient spaces X/R the quotient topology coinduced from X.

p1. Let $\mathcal{B} = \{B^2(x,r)\setminus F \subset \mathbb{R}^2 \colon x\in\mathbb{R}^2,\ r>0,\ F\subset\mathbb{R}^2 \text{ finite}\}$ be a collection of subsets of $\mathbb{R}^2$, where $B^2((x_1,x_2),r) = \{(y_1,y_2)\in\mathbb{R}^2 \colon (x_1-y_1)^2+(x_2-y_2)^2<r^2\}$ is the Euclidean ball of radius $r>0$ having center $(x_1,x_2)\in\mathbb{R}^2$.

(a) (3p) Show that $\mathcal{B}$ is a basis of a topology, denoted $\mathcal{T}_{\mathcal{B}}$, of $\mathbb{R}^2$.

(b) (3p) Show that $\mathcal{T}_{\mathcal{B}}$ is the standard Euclidean topology on $\mathbb{R}^2$.

Solution:

(a) Since $\mathcal{B}$ contains every Euclidean ball $B^2(x,r)$, we have that $\bigcup\mathcal{B} = \mathbb{R}^2$. Let now $B_1,B_2\in\mathcal{B}$ and $x\in B_1\cap B_2$. Then $B_1 = B^2(x_1,r_1)\setminus F_1$ and $B_2 = B^2(x_2,r_2)\setminus F_2$. Where F_1 and F_2 are finite. Since $x\in B_1\cap B_2$ and open Euclidean balls form a basis of the Euclidean topology, there exists $r>0$ for which $B^2(x,r)\subset B^2(x_1,r_1)\cap B^2(x_2,r_2)$. Now $B = B^2(x,r)\setminus(F_1\cup F_2)\in\mathcal{B}$ is contained in $B_1\cap B_2$. Thus $\mathcal{B}$ is a basis by the basis theorem.

(b) Since each Euclidean ball is contained in $\mathcal{B}$, we have that $\mathcal{T}_{\mathbb{R}^2}\subset\mathcal{T}_{\mathcal{B}}$. Similarly, let $B\in\mathcal{B}$. Since finite sets in $\mathbb{R}^2$ are closed in the Euclidean topology, we have that B is an open set in $\mathcal{T}_{\mathbb{R}^2}$. Thus $\mathcal{T}_{\mathcal{B}}\subset\mathcal{T}_{\mathbb{R}^2}$.

p2. (6p) Let X be a topological space and let R be an equivalence relation on X having the property that equivalence classes of R are closed sets. Show that the quotient space X/R is a T_1 space.

Solution: Solution 1: Let $q_R\colon X\to X/R$ be the quotient map. Let $a,b\in X/R$, $a\neq b$. Let also $U_a = (X/R)\setminus\{b\}$ and $U_b = (X/R)\setminus\{a\}$. Then $q_R^{-1}U_a = X\setminus q_R^{-1}(b)$ and $q_R^{-1}U_b = X\setminus q_R^{-1}(a)$. Since $q_R^{-1}(a)$ and $q_R^{-1}(b)$ are equivalence classes of R, we have that $q_R^{-1}U_a$ and $q_R^{-1}U_b$ are open sets. Thus, by definition of quotient topology, U_a and U_b are open sets and, a fortiori, neighborhoods of a and b, respectively. Thus X/R is a T_1 space.

Solution 2: (using characterization of T_1 spaces) It suffices to show that singletons in X/R are closed sets. Let $q_R\colon X\to X/R$ be the quotient map. Let $c\in X/R$. Then, by definition of the quotient map, $q_R^{-1}(c)$ is an equivalence class of R. Hence $q_R^{-1}(c)$ is a closed set in X. Thus, $\{c\}$ is a closed set in X/R. Since singletons $\{c\}$ for $c\in X/R$ are closed in X/R, the quotient space X/R is T_1.

p3. (6p) Let X and Y be topological spaces, Z the product space $Z = X\times Y$, and I a set. Show that product spaces $X^I\times Y^I$ and Z^I are homeomorphic.

Solution: Let $f\colon X^I\times Y^I\to(X\times Y)^I$ be the map $((x_i)_{i\in I},(y_i)_{i\in I})\mapsto(x_i,y_i)_{i\in I}$. Then the map $g\colon(X\times Y)^I\to X^I\times Y^I$, $(x_i,y_i)_{i\in I}\mapsto((x_i)_{i\in I},(y_i)_{i\in I})$, is the inverse of f. Thus it suffices to show that f and g are continuous.

Let $\mathrm{pr}_j^C \colon C^I \to C$, $(c_i)_{i\in I} \mapsto c_j$, be the projection, where C is any of the spaces X, Y, and Z.

To show that f is continuous, we observe first that, for any $(x,y) \in X^I \times Y^I$, where $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$, we have that $(\mathrm{pr}_j^Z \circ f)(x,y) = (x_j, y_j) = (\mathrm{pr}_j^X(x), \mathrm{pr}_j^Y(y))$. Since $\mathrm{pr}_j^X \colon X^I \to X$ and $\mathrm{pr}_j^Y \colon Y^I \to Y$ are continuous and they are the coordinate maps of $\mathrm{pr}_j^Z \circ f \colon X^I \times Y^I \to X \times Y$, we conclude that $\mathrm{pr}_j^Z \circ f$ is continuous for each $j \in I$. Since the coordinate maps of f are continuous, so is f.

To show that g is continuous, let $\mathrm{pr}_X \colon X \times Y \to X$, $(x,y) \mapsto x$, and $\mathrm{pr}_Y \colon X \times Y \to Y$, $(x,y) \mapsto y$, be projections. Let also $g_X \colon (X \times Y)^I \to X^I$, $(x_i, y_i)_{i\in I} \mapsto (x_i)_{i\in I}$, and $g_Y \colon (X \times Y)^I \to Y^I$, $(x_i, y_i)_{i\in I} \mapsto (y_i)_{i\in I}$ be the coordinate functions of g. Then, for $(x_i, y_i)_{i\in I} \in Z^I$ and $j \in I$, $(\mathrm{pr}_j^X \circ g_X)((x_i, y_i)_{i\in I}) = x_j = (\mathrm{pr}_X \circ \mathrm{pr}_j^Z)((x_i, y_i)_{i\in I})$ and $(\mathrm{pr}_j^Y \circ g_Y)((x_i, y_i)_{i\in I}) = (\mathrm{pr}_Y \circ \mathrm{pr}_j^Z)(x_i, y_i))_{i\in I}$. Thus $\mathrm{pr}_j^X \circ g_X$ and $\mathrm{pr}_j^Y \circ g_Y$ are continuous for each $j \in I$. Since coordinate functions of g_X and g_Y are continuous, so are g_X and g_Y. Hence also g is continuous.

p4. (6p) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{R}\setminus\mathbb{Q}$ having the property that $\mathbb{R}\setminus\mathbb{Q} = \bigcup_{n\in\mathbb{N}} A_n$. Show that there exists $q \in \mathbb{Q}$ and $m \in \mathbb{N}$ having the property that q is an accumulation point of A_m. (*Hint:* Baire's theorem for closed sets.)

Solution: Let $\{q_n \in \mathbb{Q} \colon n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$. For each $n \in \mathbb{N}$, let $F_{2n} = \overline{A_n}$ and $F_{2n-1} = \{q_n\}$. Then $\mathbb{R} = \bigcup_{n\in\mathbb{N}} F_n$. Since each F_n is closed, we have, by Baire's theorem, that there exists $n_0 \in \mathbb{N}$ for which $\mathrm{int} F_{n_0} \neq \emptyset$. Since each F_{2n-1} has empty intererior, we have that n_0 is even and $\mathrm{int}\overline{A_{n_0/2}} \neq \emptyset$. Let $m = n_0/2$ and $q \in \mathrm{int}\overline{A_m} \cap \mathbb{Q}$. Let U be a neighborhood of q. Since $q \in \overline{A_m}$, we have that $U \cap A_m \neq \emptyset$. Since $q \in \mathbb{Q}$ and $A_m \subset \mathbb{R}\setminus\mathbb{Q}$, we have that $(U \setminus \{q\}) \cap A_m \neq \emptyset$. Thus q is an accumulation point of A_m.