Topology II
Exam Thursday October 23, 2025
Exam time 14.00-17.00

In the following problems $\mathbb{R}^n$ is given the Euclidean topology unless otherwise specified. Product spaces $\prod_{i\in I} X_i$ are given the product topology induced from spaces X_i.

p1. Let $\mathcal{B} = \{(]a,b[\cap\mathbb{Q}) \cup (\mathbb{R}\setminus\mathbb{Q}) \subset \mathbb{R} \colon a<b\}$ be a collection of subsets of $\mathbb{R}$.

(a) (3p) Show that $\mathcal{B}$ is a basis of a topology, denoted $\mathcal{T}_{\mathcal{B}}$, of $\mathbb{R}$.

(b) (3p) Show that $A = [0,1] \subset \mathbb{R}$ is dense in space $(\mathbb{R}, \mathcal{T}_{\mathcal{B}})$.

Problem statement has a typo: In the definition of the set $\mathcal{B}$ condition $a<b$ should be replaced with $a \le b$ or $a,b\in\mathbb{R}$, that is, set $\mathcal{B}$ should read

$$\mathcal{B} = \{(]a,b[\cap\mathbb{Q}) \cup (\mathbb{R}\setminus\mathbb{Q}) \subset \mathbb{R} \colon a,b\in\mathbb{R}\},$$

where $]a,b[=\emptyset$ for $a\ge b$. This error has been taken into account – towards students benefit – in the grading of the answers to this problem.

Solution:

(a) Since $\mathbb{R} = \bigcup_{n\in\mathbb{N}}(]-n,n[\cap\mathbb{Q}) \cup (\mathbb{R}\setminus\mathbb{Q})$, we have that $\bigcup\mathcal{B} = \mathbb{R}$. Let now $x\in\mathbb{R}$ and $B_1, B_2 \in \mathcal{B}$ for which $x \in B_1\cap B_2$. Then $B_1 = (]a_1,b_1[\cap\mathbb{Q}) \cup (\mathbb{R}\setminus\mathbb{Q})$ and $B_2 = (]a_2,b_2[\cap\mathbb{Q}) \cup (\mathbb{R}\setminus\mathbb{Q})$. Thus $B_1\cap B_2 = ((]a_1,b_1[\cap]a_2,b_2[)\cap\mathbb{Q}) \cup (\mathbb{R}\setminus\mathbb{Q})$. Since $x\in B_1\cap B_2$, we have that $]a_1,b_1[\cap]a_2,b_2[$ is a **non-empty** interval **or the empty set**. In both cases, $B_1\cap B_2 \in \mathcal{B}$. Thus $\mathcal{B}$ is a basis of a topology by the basis theorem.

(b) Let $x\in\mathbb{R}$ and U a neighborhood of x in $(\mathbb{R},\mathcal{T}_{\mathcal{B}})$. Then exists $B\in\mathcal{B}$ for which $x\in B\subset U$. Since $[0,1]\cap(\mathbb{R}\setminus\mathbb{Q}) \ne \emptyset$, we have that $B\cap[0,1]\ne\emptyset$. Thus $U\cap[0,1]\ne\emptyset$. Hence x is in the closure of $[0,1]$ and $[0,1]$ is dense in $(\mathbb{R},\mathcal{T}_{\mathcal{B}})$.

p2. (6p) Let X be a set and let $E\subset X$ and $F\subset X$ be subsets. Suppose that $(E,\mathcal{T}_E)$ and $(F,\mathcal{T}_F)$ are T_1 spaces. Let $\mathcal{T}$ be a topology on X coinduced by inclusions $\iota_E = \mathrm{id}|_E \colon E, \to X$ and $\iota_F = \mathrm{id}|_F \colon F\to X$ from $(E,\mathcal{T}_E)$ and $(F,\mathcal{T}_F)$, respectively. Show that $(X,\mathcal{T})$ is a T_1 space.

Solution: Let $x\in X$. If $x\in E$, we have that $\iota_E^{-1}(x) = \{x\}$ is a closed set in $(E,\mathcal{T}_E)$. If $x\in F$, we similarly have that $\{x\}$ is a closed in $(F,\mathcal{T}_F)$. Thus, if $x\in E\cup F$, $\{x\}$ closed in X. Suppose now that $x\notin E\cup F$. Then $\iota_E^{-1}\{x\} = \iota_F^{-1}\{x\} = \emptyset$. Thus, again, $\{x\}$ is closed in X. We conclude that $(X,\mathcal{T})$ is a T_1 space.

p3. (6p) Suppose there exists an embedding of a topological space X to a topological space Y, and let I be a set. Show that there exists an embedding of the product space X^I into the product space Y^I.

Solution:

Method 1: Let $f\colon X\to Y$ be an embedding and let $F\colon X^I\to Y^I$ be the map $(x_i)_{i\in I}\mapsto (f(x_i))_{i\in I}$.

We show first that F is injective. Let $a = (a_i)_{i\in I}$ and $b = (b_i)_{i\in I}$ be such that $F(a) = F(b)$. Then $f(a_i) = f(b_i)$ for each $i \in I$. Since f is injective, $a_i = b_i$ for each $i \in I$. Thus $a = b$ and F is injective.

We show next that F is continuous. By the universal property of the product topology, H is continuous if and only if each composition $\mathrm{pr}_j^Y \circ F\colon X^I \to Y$ is continuous, where $\mathrm{pr}_j^Y\colon Y^I \to Y$ is the projection $(y_i)_{i\in I} \mapsto y_j$. Let $\mathrm{pr}_j^X\colon X^I \to X$ be the projection $(x_i)_{i\in I} \mapsto x_j$. Then, for each $x = (x_i)_{i\in I}$ and $j \in I$, we have that $(\mathrm{pr}_j^Y \circ F)(x) = \mathrm{pr}_j^Y((f(x_i))_{i\in I}) = f(x_j) = (f \circ \mathrm{pr}_j^X)(x)$. Thus $\mathrm{pr}_j^Y \circ F = f \circ \mathrm{pr}_j^X$. Since each $f \circ \mathrm{pr}_j^X$ is continuous, we conclude that F is continuous.

It remains to show that the corestriction $F_1\colon X^I \to F(X^I)$ has continuous inverse $F_1^{-1}\colon F(X^I) \to X^I$. Let $U \subset X^I$ be an open set. We show that $(F_1^{-1})^{-1}U = FU$ is an open set in the relative topology of FX^I. Let $y = (y_i)_{i\in I} \in FU$ and fix $x = (x_i)_{i\in I} \in F^{-1}(y)$. Then there exists a finite set $K \subset I$ and open sets $U_i \subset X$ for which $U_i = X$ for $i \notin K$ and $x \in \prod_{i\in I} U_i \subset U$. Since f is an embedding there exists for each $i \in K$ an open set $W_i \subset Y$ for which $fU_i = W_i \cap fX$. For each $i \notin K$, we fix $W_i = Y$. Then $W = \prod_{i\in I} W_i$ is an open set in Y^I and $V = W \cap FX^I$ is an open set in FX^I. Furthermore, $V = W \cap FX^I = \prod_{i\in I} W_i \cap fX = \prod_{i\in I} fU_i \subset FU$. Hence FU is an open set.

Method $1\frac{1}{2}$: Same as above, but instead of proving continuity of the inverse of the corestriction, show that the corestriction of F is an open map. For this suffices to show that F maps subbasis elements to open sets. A subbasis of the product topology of X^I is given by elements $\mathrm{pr}_j^{-1}U$, where $j \in I$ and $U \subset X$ is an open set. Then $F(\mathrm{pr}_j^{X\,-1}U) = (\mathrm{pr}_j^Y)^{-1}(fU) \cap FX^I$. Hence the corestriction of F is an open map.

Method 2: Let $f\colon X \to Y$ be an embedding and $f_!\colon X \to fX$ a corestriction of f.

Let $F\colon X^I \to (fX)^I$ be the map $(x_i)_{i\in I} \mapsto (f_!(x_i))_{i\in I}$ and $G\colon (fX)^I \to X^I$ the map $(y_i)_{i\in I} \mapsto (f_!^{-1}(y_i))_{i\in I}$. Then $G \circ F = \mathrm{id}_{X^I}$ and $F \circ G = \mathrm{id}_{(fX)^I}$.

To show that F and G are continuous, let $\mathrm{pr}_j\colon X^I \to X$, $(x_i)_{i\in I} \mapsto x_j$, and $\pi_j\colon (fX)^I \to fX$, $(y_i)_{i\in I} \mapsto y_j$, be projections. Since $\pi_j \circ F = f_! \circ \mathrm{pr}_j = f \circ \mathrm{pr}_j$ and $\mathrm{pr}_j \circ G \circ f_!^{-1} \circ \pi_j$, we have that F component maps $\pi_j \circ F$ and $\mathrm{pr}_j \circ G$ of F and G, respectively, are continuous, Thus F and G are continuous by the universal property of the product topology. Thus $F\colon X^I \to (fX)^I$ is a homeomorphism. Since the product topology of $(fX)^I$ is the subspace topology induced from Y^I, we conclude that F is an embedding.

p4. (6p) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{R}\setminus\mathbb{Q}$ having the property that $\mathbb{R}\setminus\mathbb{Q} = \bigcup_{n\in\mathbb{N}} A_n$. Let $N = \{x \in \mathbb{R}\colon x \notin \overline{A_n} \text{ for every } n \in \mathbb{N}\}$. Show that N has empty interior in $\mathbb{R}$. (*Hint:* Baire's theorem for closed sets.)

Solution:

Method 1: Suppose towards contradiction that the set N has non-empty interior $U = \mathrm{int}N$ and let $[a,b] \subset U$ be an interval.

Let $\{q_n \in \mathbb{Q}\colon n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}$. For each $n \in \mathbb{N}$, let $F_{2n} = \overline{A_n} \cap [a,b]$ and $F_{2n-1} = \{q_n\} \cap [a,b]$. Then $\bigcup_{n\in\mathbb{N}} F_n = [a,b]$. Since $[a,b]$ is a complete metric space and each F_n is closed, we have, by Baire's theorem, that there exists $n_0 \in \mathbb{N}$ for which $\mathrm{int}F_{n_0} \neq \emptyset$. Since each F_{2n-1} has empty interior, we have that n_0 is even and $\mathrm{int}\left(\overline{A_{n_0/2}} \cap [a,b]\right) \neq \emptyset$. Let $m = n_0/2$. Let now $x \in \mathrm{int}\left(\overline{A_m} \cap [a,b]\right)$. Then $x \in \overline{A_m}$ and $x \in [a,b] \subset U \subset N$. This contradicts the definition of N. Thus N has empty interior.

Method 2: (easy proof) Since $\mathbb{R} \setminus N = \bigcup_{n \in \mathbb{N}} \overline{A_n} \supset \mathbb{R} \setminus \mathbb{Q}$, we have that $N \subset \mathbb{Q}$. Thus N has no interior in $\mathbb{R}$.