Topolgy IB: Practice exam

Length of exam: 2h 30min. Aiding material (including calculators) is NOT allowed in the exam.

Problem 1

(a) State the definition of a bilipschitz map. (2p.)

(b) Show that the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = (2x,y) \quad \text{for all } (x,y) \in \mathbb{R}^2$$

is bilipschitz. Use the standard Euclidean metric in $\mathbb{R}^2$. (4p.)

Problem 2

(a) State the definition of a complete metric space. (2p.)

(b) Let (X,d) be a nonempty complete metric space. Suppose that $A \subset X$ is complete. (That is, (A,d_A) is complete, where the metric $d_A\colon A \times A \to \mathbb{R}$ is given by $d_A(x,y) = d(x,y)$ for all $(x,y) \in A \times A$.) Show that A is closed in X. (4p.)

Problem 3

(a) State the definition of a compact metric space. (2p.)

(b) Let $A \subset \mathbb{R}^2$ be a non-empty closed and bounded subset in the standard Euclidean metric. Show that there are points $(c_1,c_2) \in A$ and $(d_1,d_2) \in A$ such that

$$x_1 \leq c_1 \quad \text{and} \quad d_2 \leq x_2$$

for all $(x_1,x_2) \in A$. (Hint: Min-max theorem applied to suitable continuous maps $\mathbb{R}^2 \to \mathbb{R}$.) (4p.)

Problem 4

(a) State the definition of a path-connected metric space. (2p.)

(b) Consider the metric space

$$\mathbb{R}^2 \setminus \mathbb{Q}^2 = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R} \setminus \mathbb{Q} \quad \text{or} \quad y \in \mathbb{R} \setminus \mathbb{Q}\}.$$

with the standard Euclidean metric. Is $\mathbb{R}^2 \setminus \mathbb{Q}^2$ path-connected? Please explain your argument carefully. (Hint: Draw a picture.) (4p.)