Topology IA- Exam 9/3/2023

1. (a) (2p) Show that the function $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, defines a metric in $\mathbb{R}^n$.
   (b) (2p) Examine whether or not this metric comes from a norm. If yes, find this norm and prove that it is indeed a norm in $\mathbb{R}^n$.
   (c) (1p) Using the parallelogram law $|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$, which is true for every inner product space with $|\cdot|$ being the induced norm, examine whether or not the norm from (b) comes from an inner product.

2. (a) (2p) Let (X,d) be metric space and $f, g : (X,d) \to (\mathbb{R}, e)$ continuous functions from X to $\mathbb{R}$ equipped with the euclidean metric. Prove that the function $f+g$ is continuous.
   (b) (1.5p) Explain why the set $\{x \in X : f(x) < g(x)\}$ is open in (X,d).
   (c) (1.5p) Show that the set $A = \{(x,y) \in \mathbb{R}^2 : \cos x \leq y \leq e^{x+y}\}$ is a closed subset of $\mathbb{R}^2$.

3. (a) (3p) Find the interior,exterior, closure and boundary of the set
$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x > 0\}.$$
   (b) (2p) If (X,d) and (Y,d') are metric spaces show that $f : X \to Y$ is continuous if and only if for all $B \subseteq Y$ we have that
$$f(\overline{B}) \subseteq \overline{f(B)}.$$

4. (a) (2p) Give an example of a metric space (X,d) in which $\overline{B(x,r)} \neq \overline{B}(x,r)$, for some $x \in X$ and $r > 0$.
   (b) (2p) Give an example of a set A in a metric space (X,d) which is both open and closed but it is not the empty set or the entire space.
   (c) (1p) Give an example of a continuous function between two metric spaces X and Y that does not map closed sets to closed sets.

Notes, calculators, phones etc. are not allowed in the exam.