Topics in Logic: Generalized Descriptive Set Theory
Exam

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We will denote by κ^κ the set of functions $f : \kappa \to \kappa$, and $\kappa^{<\kappa}$ the set of functions $f : \alpha \to \kappa$ where $\alpha < \kappa$. Recall that κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

Q. 1. We say that a tree $T \subseteq \theta^{<\kappa}$ is pruned if for all $f \in T$ and $\beta > \alpha = dom(f)$, there is $g \in T$ such that $f = g \restriction \alpha$ and $\beta = dom(g)$. We define the body of a pruned tree T as the set

$$[T] = \{\eta \in \theta^\kappa \mid \forall \alpha < \kappa,\ \eta \restriction \alpha \in T\}.$$

Question. (20 points). Show that $A \subseteq \kappa^\kappa$ is closed if and only if there is a pruned tree of $\kappa^{<\kappa}$ such that $[T] = A$.

Q. 2.

- A function $f\colon \kappa^\kappa \to \kappa^\kappa$ is said to be *κ-Borel* if for any open set $A \subseteq \theta^\kappa$, $f^{-1}[A]$ is a κ-Borel set.
- A tree T is a κ^+, λ-tree if does not contain chains of length λ and its cardinality is less than κ^+. It is closed if every chain has a unique supremum in T.
- A pair (T, h) is a *κ-Borel**-code if T is a closed κ^+, λ-tree, $\lambda \leq \kappa$, and h is a function with domain T such that if $x \in T$ is a leaf, then $h(x)$ is a basic open set and otherwise $h(x) \in \{\cup, \cap\}$.
- For an element $\eta \in \kappa^\kappa$ and a κ-*Borel**-code (T, h), the κ-*Borel**-game $B^*(T, h, \eta)$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T. At each move, if the game is at node $x \in T$ and $h(x) = \cap$, then **I** chooses an immediate successor y of x and the game continues from this y. If $h(x) = \cup$, then **II** makes the choice. At limits the game continues from the (unique) supremum of the previous moves. Finally, if $h(x)$ is a basic open set, then the game ends, and **II** wins if and only if $\eta \in h(x)$.
- A set $X \subseteq \kappa^\kappa$ is a *κ-Borel*-set* if there is a κ-*Borel**-code (T, h) such that for all $\eta \in \kappa^\kappa$, $\eta \in X$ if and only if **II** has a winning strategy in the game $B^*(T, h, \eta)$.

Question. (20 points). Show that if $f : \kappa^\kappa \to \kappa^\kappa$ is a κ-Borel function, then for all κ-Borel* set $B \subseteq \kappa^\kappa$, $f^{-1}[B]$ is a κ-Borel* set.

Q. 3.

- $C \subseteq \kappa$ is an unbounded set if for all $\alpha \in \kappa$ there is $\beta \in S$ such that $\alpha < \beta$.
- A set $C \subseteq \kappa$ is a club on κ if C is unbounded and any sequence $\langle \alpha_i \mid i < \gamma \rangle$ such that $\gamma < \kappa$ and for all $\alpha_i \in X$, satisfies $\bigcup_{i<\gamma} \alpha_i \in C$.
- Let $S \subseteq \kappa$ be an unbounded set (i.e. for all $\alpha \in \kappa$ there is $\beta \in S$ such that $\alpha < \beta$). We say that a function $f : \kappa^\kappa \to \kappa^\kappa$ is S-recursive if there is a function $H : \kappa^{<\kappa} \to \kappa^{<\kappa}$ such that for all $\alpha \in S$ and $\eta \in \kappa^\kappa$, $f(\eta)(\theta) = H(\eta \restriction \alpha)(\theta)$ for all $\theta < min(S \backslash (\alpha + 1))$.
- For all $\eta, \xi \in \kappa^\kappa$, denote

$$\Delta(\eta, \xi) := \min(\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\} \cup \{\kappa\}).$$

A function $f\colon \kappa^\kappa \to \kappa^\kappa$ is said to be *Lipschitz* if for all $\eta, \xi \in \kappa^\kappa$,

$$\Delta(\eta, \xi) \leq \Delta(f(\eta), f(\xi)).$$

Question.

1. (10 points). Prove that: If C is a club and $f : \kappa^\kappa \to \kappa^\kappa$ is a S-recursive function such that:
   (†) *$\alpha_m = min(S)$ is such that for all $\eta, \xi \in \kappa^\kappa$ and $\beta < \alpha_m$, $\eta \restriction \beta = \xi \restriction \beta$ implies $f(\eta) \restriction \beta = f(\xi) \restriction \beta$.*
   Then f is Lipschitz.
2. (10 points.) Find $S \subseteq \kappa$ and a function f, such that f is S-recursive but not κ-recursive.

Q. 4. – $\mathcal{F} \subseteq \mathcal{P}(\delta)$ is a filter over δ if the following holds:

* $\delta \in \mathcal{F}$,
* for all $x \in \mathcal{F}$, if $x \subseteq y$, then $y \in \mathcal{F}$,
* if $x, y \in \mathcal{F}$, then $x \cap y \in \mathcal{F}$.

– Let $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ be a filter over κ.

$$\eta \sim^{\kappa}_{\mathcal{F}} \xi \Leftrightarrow \exists W \in \mathcal{F} \ (W \subseteq \{\alpha < \kappa \mid \eta(\alpha) = \xi(\alpha)\})$$

We say that an equivalence relation E is filtered if and only if there is a filter $\mathcal{F}$ such that

$$\eta \, E \, \xi \Leftrightarrow \eta \sim^{\kappa}_{\mathcal{F}} \xi.$$

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$$E_0^{<\kappa,\kappa} := \{(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \mid \exists \alpha < \kappa \ \ \forall \beta > \alpha \ \ \eta(\beta) = \xi(\beta)\}.$$

– We denote by $E_1 \hookrightarrow_L E_2$ if thee is a Lipschitz function f such that

$$(\eta, \xi) \in E_1 \iff (f(\eta), f(\xi)) \in E_2.$$

– $S \subseteq \kappa$ is a stationary set if for every club $C \subseteq \kappa$, $C \cap S \neq \emptyset$.

Question.

1. (10 points). Show that $E_0^{<\kappa,\kappa}$ is a filtered equivalence relation.
2. (10 points). Prove that for any stationary set $S \subseteq \kappa$, $E_0^{<\kappa,\kappa} \hookrightarrow_L =^{\kappa}_{S}$.

Q. 5. – Given a filter $\mathcal{F}$ over δ, we denote by $\mathcal{F}^+$ the set $\{A \subseteq \delta \mid \forall B \in \mathcal{F}(A \cap B \neq \emptyset)\}$.

– Suppose $S \subseteq \kappa$ is a stationary set and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence of filters, i.e. for all $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α. We say that $\vec{\mathcal{F}}$ captures clubs if and only if for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is non-stationary.

– Suppose $X, S \subseteq \kappa$ are stationary sets, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence of filters. We say that X $\vec{\mathcal{F}}$-reflects to S if and only if $\vec{\mathcal{F}}$ captures clubs, and for every stationary set $Y \subseteq X$, the set $\{\alpha \in S \mid Y \cap \alpha \in \mathcal{F}_\alpha^+\}$ is stationary.

– We say that X $\mathfrak{f}$-reflects to S if and only if there exists a sequence $\vec{\mathcal{F}}$ over a stationary subset $S' \subseteq S$ such that X $\vec{\mathcal{F}}$-reflects to S'.

Question. (20 points). Prove that if X $\mathfrak{f}$-reflects to S, then $=^{\kappa}_{X} \hookrightarrow_L =^{\kappa}_{S}$.

Q. 6. – T is a classifiable theory if for any $\mathcal{A}$ and $\mathcal{B}$ models of T with domain κ,

$$\text{II} \uparrow \text{EF}^{\kappa}_{\omega}(\mathcal{A}, \mathcal{B}) \iff \mathcal{A} \cong \mathcal{B}$$

$$\text{I} \uparrow \text{EF}^{\kappa}_{\omega}(\mathcal{A}, \mathcal{B}) \iff \mathcal{A} \ncong \mathcal{B}.$$

– $\diamondsuit_\kappa(S)$ is the statement that there is a sequence $\langle D_\alpha \mid \alpha \in S \rangle$ such that

* For all $\alpha \in S$, $D_\alpha \subseteq \alpha$.
* For all $A \subseteq \lambda$, the set $\{\alpha \in S \mid D_\alpha = A \cap \alpha\}$ is stationary.

Question. (20 points). Assume T is a countable complete classifiable theory over a countable vocabulary. Let $S \subseteq \kappa$ a stationary set. If $\diamondsuit_\kappa(S)$ holds, then $\cong^{2}_{T} \hookrightarrow_L =^{2}_{S}$.