MAT12003/AYMAT12003 Probability I

Course exam 6.3.2020 – Solutions and grading

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1 Write

B_i = "ith ball is black", $i=1,2$,

W_i = "ith ball is white", $i=1,2$,

Then $\{B_1, W_1\}$ is a partition and

$P(B_1) = \frac{3}{4}$ and $P(W_1) = \frac{1}{4}$. (+1p)

Moreover, $P(B_2 | B_1) = \frac{1}{2}$ and $P(B_2 | W_1) = 1$.
(+1p)

(a) Using the law of total probability

$$P(B_2) = P(B_2|B_1)P(B_1) + P(B_2|W_1)P(W_1) \quad (+1p)$$

$$= \frac{1}{2} \cdot \frac{3}{4} + 1 \cdot \frac{1}{4} = \frac{3}{8} + \frac{1}{4} = \underline{\underline{\frac{5}{8}}} = 0{,}625 \quad (+1p)$$

(b) Using the Bayes formula

$$P(B_1|B_2) = \frac{P(B_2|B_1)P(B_1)}{P(B_2)} \quad (+1p)$$

$$= \frac{\frac{1}{2} \cdot \frac{3}{4}}{\frac{5}{8}} = \frac{3}{8} \cdot \frac{8}{5} = \underline{\underline{\frac{3}{5}}} = 0{,}6. \quad (+1p)$$

② Write

$n =$ "number of the sold tickets" $= 50$,

$p = P($"a customer does not show up"$) = \frac{1}{10}$,

$X =$ "number of the customers who do not show up for the tour". (+1p)

Way I: By using the binomial distribution:

$X \sim Bin(50, \frac{1}{10})$ (+1p)

P("everyone who shows up for the tour will have a seat in the bus")

(+1p)

$= P(X \geq 2)$

$= 1 - P(X \leq 1)$ (+1p)

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - \left[\binom{50}{0}\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{50} + \binom{50}{1}\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^{49}\right]$ (+1p)

$= 1 - \left[\left(\frac{9}{10}\right)^{50} + 50 \cdot \frac{1}{10} \cdot \left(\frac{9}{10}\right)^{49}\right]$

$\approx \underline{\underline{0{,}966}}$. (+1p)

Way II: By usin the Poisson distribution

$$X \sim \text{Poisson}\left(50 \cdot \frac{1}{10}\right) = \text{Poisson}(5)$$ (+1p)

P("everyone who shows up for the tour will have a seat in the bus")

$= P(X \geq 2)$ (+1p)

$= 1 - P(X \leq 1)$

$= 1 - \left[P(X=0) + P(X=1)\right]$ (+1p)

$= 1 - e^{-5}\left[\frac{5^0}{0!} + \frac{5^1}{1!}\right]$ (+1p)

$= 1 - e^{-5}(1+5) = 1 - 6e^{-5}$

$\approx \underline{0{,}960}$. (+1p)

3. Write

X = "number of the outcomes five and six".

Then $X \sim Bin(300, \frac{1}{3})$ and hence

$EX = 300 \cdot \frac{1}{3} = 100$ and

$D^2X = 300 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{200}{3}$, $DX = \sqrt{\frac{200}{3}}$. (+1p)

The random variable $\frac{X-100}{\sqrt{\frac{200}{3}}}$ is approximately $N(0,1)$-distributed (+1p)

therefore

$$P(X \leq 80) \overset{\text{cont. fix.}}{=} P(X \leq 80{,}5) \quad (+1p)$$

$$\overset{\text{stand.}}{=} P\left(\frac{X-100}{\sqrt{\frac{200}{3}}} \leq \frac{80{,}5-100}{\sqrt{\frac{200}{3}}}\right) \quad (+1p)$$

$$\overset{\text{norm. appr.}}{=} \Phi\left(-\frac{19{,}5}{\sqrt{\frac{200}{3}}}\right) = \Phi(-2{,}39) \quad (+1p)$$

$$= 1 - \Phi(2{,}39) = 1 - 0{,}991576$$

$$= 0{,}008424 \approx \underline{0{,}0084}. \quad (+1p)$$

4. (a) The cumulative distribution function of Y is

$$F_Y(y) = P(Y \le y) = P(|X+1| \le y)$$

$$= \begin{cases} 0, & y \le 0 \\ P(-y \le X+1 \le y), & y > 0 \end{cases} \quad (+1p)$$

$$= \begin{cases} 0, & y \le 0 \\ P(-y-1 \le X \le y-1), & y > 0 \end{cases}$$

$$= \begin{cases} 0, & y \le 0 \\ F_X(y-1) - F_X(-y-1), & y > 0 \end{cases}$$

$$= \begin{cases} 0, & y \le 0 \\ \frac{y-1+1}{2} - 0 = \frac{y}{2}, & 0 < y < 2 \\ 1, & y \ge 2 \end{cases} \quad (+1p)$$

Since the cumulative distribution function of X is

$$F_X(x) = \begin{cases} 0, & x \le -1 \\ \frac{x-(-1)}{1-(-1)} = \frac{x+1}{2}, & -1 < x < 1 \\ 1, & x \ge 1. \end{cases} \quad (+1p)$$

Therefore the density function of $\underline{Y}$ is

$$f_{\underline{Y}}(x) = F'_{\underline{Y}}(y) = \begin{cases} \frac{1}{2}, & 0<y<2. \\ 0, & \text{otherwise}. \end{cases}$$

(+1p)

(b) Way I by integration:

$$P(-1<\underline{Y}<1) = \int_{-1}^{1} f_{\underline{Y}}(y)\,dy = \int_{0}^{1} \frac{1}{2}\,dy \quad \text{(+1p)}$$

$$= \Big/_{0}^{1} \frac{1}{2}y = \frac{1}{2} - 0 = \frac{1}{2}. \quad \text{(+1p)}$$

Way II by using the cumulative distribution function:

$$P(-1<\underline{Y}<1) = F_{\underline{Y}}(1) - F_{\underline{Y}}(-1) \quad \text{(+1p)}$$

$$= \frac{1}{2} - 0 = \frac{1}{2}. \quad \text{(+1p)}$$