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Dept of Mathematics and Statistics
MAT12003/AYMAT12003
Probability I
Moodle home exam
12.3.2021 at 12:00–15:00

Return your solutions to the submission folder in MOOC by 15:00.

1. In a box there are four red balls and in addition black balls. The number of the black balls is not equal to the number of the red balls. Consider picking up two balls randomly from the box without replacement. Then the probability to get one red ball and one black ball is $4/7$.
   (a) How many black balls there are in the box?
   (b) What is the probability to get red balls only when picking up two balls randomly from the box?
   (c) What is the expectation of the number of red balls in picking up two balls?

2. Suppose that the events $A_1, \dots, A_n$ form a partition of the sample space Ω and $P(A_i) > 0$ for all $i = 1, \dots, n$. Prove that if B is an event and $P(B \mid A_i) = p$ for all $i = 1, \dots, n$, then $p = P(B)$.

3. In a mathematics course there are six exercise sets and six assignments in each exercise set. All assignmets are on average equally demanding and solving them is independent. The solving probability of every single assignment for a student A is supposed to be $\frac{5}{6}$.
   (a) Give a random variable X which suits for the situation. Find the probability that A solves at least 35 assignments during the course.
   (b) Use normal approximation to find the probability that A solves at least 20 but at most 30 assignments during the course.

4. Let n be a positive integer. A random variable X has a continuous distribution with the probability density function $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} cx^n, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the constant c.
   (b) Derive the probability density function of the transformation $Y = \sqrt{X}$.

A table of the values of the cumulative distribution function of the standard normal distribution and a list of frequency and density functions, expectations and variances of some distributions are given in the next page.

Values of the cumulative distribution function Φ of the standard normal distribution; $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} dt$

x	0,00	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,500000	0,503989	0,507978	0,511966	0,515953	0,519938	0,523922	0,527903	0,531881	0,535856
0,1	0,539828	0,543795	0,547758	0,551717	0,555670	0,559618	0,563560	0,567495	0,571424	0,575345
0,2	0,579260	0,583166	0,587064	0,590954	0,594835	0,598706	0,602568	0,606420	0,610261	0,614092
0,3	0,617911	0,621720	0,625616	0,629300	0,633072	0,636831	0,640576	0,644309	0,648027	0,651732
0,4	0,655422	0,659097	0,662757	0,666402	0,670031	0,673645	0,677242	0,680822	0,684386	0,687933
0,5	0,691462	0,694974	0,698468	0,702944	0,705402	0,708840	0,712260	0,715661	0,719043	0,722405
0,6	0,725747	0,729069	0,732371	0,735653	0,738914	0,742154	0,745373	0,748571	0,751748	0,754903
0,7	0,758036	0,761148	0,764238	0,767305	0,770350	0,773373	0,776373	0,779350	0,782305	0,785236
0,8	0,788145	0,791030	0,793892	0,796731	0,799546	0,802338	0,805106	0,807850	0,810570	0,813267
0,9	0,815940	0,818589	0,821214	0,823814	0,826391	0,828944	0,831472	0,833977	0,836457	0,838913
1,0	0,841345	0,843752	0,846136	0,848495	0,850830	0,853141	0,855428	0,857690	0,859929	0,862143
1,1	0,864334	0,866500	0,868643	0,870762	0,872857	0,874928	0,876976	0,879000	0,881000	0,882977
1,2	0,884930	0,886861	0,888768	0,890651	0,892512	0,894350	0,896165	0,897958	0,899727	0,901475
1,3	0,903200	0,904902	0,906582	0,908241	0,909877	0,911492	0,913085	0,914656	0,916207	0,917736
1,4	0,919243	0,920730	0,922196	0,923642	0,925066	0,926471	0,927855	0,929219	0,930563	0,931889
1,5	0,933193	0,934478	0,935744	0,936992	0,938220	0,939429	0,940620	0,941792	0,942947	0,944083
1,6	0,945201	0,946301	0,947384	0,948449	0,949497	0,950528	0,951543	0,952540	0,953521	0,954486
1,7	0,955434	0,956367	0,957284	0,958185	0,959070	0,959941	0,960796	0,961636	0,962462	0,963273
1,8	0,964070	0,964852	0,965620	0,966375	0,967116	0,967843	0,968557	0,969258	0,969946	0,970621
1,9	0,971283	0,971933	0,972571	0,973197	0,973810	0,974412	0,975002	0,975581	0,976148	0,976704
2,0	0,977250	0,977784	0,978308	0,978822	0,979325	0,979818	0,980301	0,980774	0,981237	0,981691
2,1	0,982136	0,982571	0,982997	0,983414	0,983823	0,984222	0,984614	0,984997	0,985371	0,985738
2,2	0,986097	0,986447	0,986791	0,987126	0,987454	0,987776	0,988089	0,988396	0,988696	0,988989
2,3	0,989276	0,989556	0,989830	0,990097	0,990358	0,990613	0,990862	0,991106	0,991344	0,991576
2,4	0,991802	0,992024	0,992240	0,992451	0,992656	0,992857	0,993053	0,993244	0,993431	0,993613
2,5	0,993790	0,993963	0,994132	0,994297	0,994457	0,994614	0,994766	0,994915	0,995060	0,995201
2,6	0,995339	0,995473	0,995604	0,995731	0,995855	0,995975	0,996093	0,996207	0,996319	0,996427
2,7	0,996533	0,996636	0,996736	0,996833	0,996928	0,997020	0,997110	0,997197	0,997282	0,997365
2,8	0,997445	0,997523	0,997599	0,997673	0,997744	0,997814	0,997882	0,997948	0,998012	0,998074
2,9	0,998134	0,998193	0,998250	0,998305	0,998359	0,998411	0,998462	0,998511	0,998559	0,998605
	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
3,0	0,998650	0,999032	0,999313	0,999517	0,999663	0,999767	0,999841	0,999892	0,999928	0,999952

Frequency and density functions, expectations and variances of distributions

$$X \sim \text{Bernoulli}(p) \implies P(X=k) = p^k(1-p)^{1-k},\ k = 0,1;$$
$$EX = p \text{ and } D^2X = p(1-p).$$

$$X \sim \text{Bin}(n,p) \implies P(X=k) = \binom{n}{k} p^k (1-p)^{n-k},\ k = 0,1,\dots,n;$$
$$EX = np \text{ and } D^2X = np(1-p).$$

$$X \sim \text{Hyperg}(N,K,n) \implies P(X=k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}},\ k = 0,1,\dots,n;$$
$$EX = n\frac{K}{N} \text{ and } D^2X = n\frac{K}{N}\frac{N-K}{N}\frac{N-n}{N-1}.$$

$$X \sim \text{Geom}(p) \implies P(X=k) = p(1-p)^k,\ k = 0,1,2,\dots;$$
$$EX = \frac{1-p}{p} \text{ and } D^2X = \frac{1-p}{p^2}.$$

$$X \sim \text{Poisson}(\lambda) \implies P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!},\ k = 0,1,2,\dots;\ EX = \lambda \text{ and } D^2X = \lambda.$$

$$X \sim \text{Uni}(a,b) \implies f(x) = \begin{cases} \frac{1}{b-a}, & x \in (a,b), \\ 0, & \text{otherwise}; \end{cases} \quad EX = \frac{a+b}{2} \text{ and } D^2X = \frac{(b-a)^2}{12}.$$

$$X \sim \text{Exp}(\lambda) \implies f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise}; \end{cases} \quad EX = \frac{1}{\lambda} \text{ and } D^2X = \frac{1}{\lambda^2}.$$

$$X \sim N(\mu,\sigma^2) \implies f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},\ x \in \mathbb{R};\ EX = \mu \text{ and } D^2X = \sigma^2.$$