Stochastic analysis II, MAST31710 Fall 2022, Home Exam

To be returned by March 3rd 2023. You can use the course materials and any book or paper you may find relevant, you can also collaborate with fellow students, and ask the teacher for explanations and hints. You do not need solve all the problems, it's enough to answer 2/3 of the questions to pass with the best grades, choose those which you find most interesting.

1. **From Ito formula to Wick formula for joint Gaussian moments**

   Let $(W_t^{(1)}, \dots, W_t^{(n)} : t \geq 0)$ be correlated Brownian motions with $W_0^{(j)} = 0$ with covariance

   $$\mathbb{E}\big(W_t^{(j)} W_t^{(k)}\big) = \langle W^{(j)}, W^{(k)} \rangle_t = c_{jk}\, t, \quad t \geq 0$$

   where (c_{jk}) is a positive definite (covariance) matrix with $c_{jj} = 1$

   Use inductively Ito formula and Fubini Theorem to obtain the following *Wick formula* for the joint moments at time t:

   $$E_P\big(W_t^{(1)} \dots W_t^{(n)}\big) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ t^{n/2} \sum\limits_{\text{pairings}} \prod\limits_{pairs\{i,j\}} c_{ij} & \text{if } n \text{ is even} \end{cases}$$

   where when n is even, the sum is over all pairings of $1, \dots, n$ into $n/2$ pairs, where the pairs are disjoint and the elements of the pairs are distinct, and for each pairing we then take the product over the pairs of the pairing.

   This is Wick's formula for zero mean jointy Gaussian random variables (in the literature and in the lecture notes you can find a proof based on the moment generating function).

   Hints: Use the integration by parts formula for semimartingales or Ito formula to compute the semimartingale decomposition of the product $W_t^{(1)} \dots W_t^{(n)}$, and show that the local martingale part has to be a true martingale with zero expectation.

   Hint: Let's see how Wick formula works in practice, for example

   $$E_P\big(W_t^{(1)} W_t^{(2)} W_t^{(3)} W_t^{(4)}\big) = (c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23})t^2$$

   since we can form disjoint pairs in three possible way, and each pairing contributes with the product of two covariances.

   Another example would be

   $$E_P\big((W_t^{(1)})^2 W_t^{(2)} W_t^{(3)}\big) = (c_{11}c_{23} + 2c_{12}c_{13})t^2$$

   and

   $$E_P\big((W_t^{(1)})^2 (W_t^{(2)})^2\big) = (c_{11}c_{22} + 2c_{12}^2)t^2$$

2. **Stochastic exponential and linear stochastic differential equations**

   Let $X(t) = X(0) + M(t) + A(t)$ be a continuous semimartingale, where $M(t)$ is a local martingale and $A(t)$ is adapted with finite variation.

Doleans **stochastic exponential** is defined as

$$\mathcal{E}_t(X) = \exp\left(X(t) - \frac{\langle X\rangle_t}{2}\right)$$

where $\langle X\rangle_t = \langle M\rangle_t$.

(a) What is the predictable variation of the Doleans exponential $\langle \mathcal{E}(X)\rangle_t$?

(b) Use Ito formula to show that $\mathcal{E}_t(X)$ solves the homogeneous linear stochastic differential equation

$$Z(t) = 1 + \int_0^t Z(s)dX(s)$$

where the integral is an Ito integral.

(c) Let $Y(t)$ be another semimartingale.
Find the solution of the non-homogeneous linear stochastic differential equation

$$Z(t) = 1 + Y(t) + \int_0^t Z(s)dX(s)$$

Hint: try a solution of product form $Z(t) = \mathcal{E}_t(X)U(t)$ for a suitable semimartingale $U(t)$.
Hint. Use the integration by parts formula of Ito calculus.

(d) Express the product

$$\mathcal{E}_t(X)\mathcal{E}_t(Y)$$

as the stochastic exponential $\mathcal{E}_t(V)$ of a suitable semimartingale $V(t)$.
Hint. Use the integration by parts formula of Ito calculus.

3. (Bougerol' identity) Consider
$\sinh(W(t))$ where $W(t)$ is Brownian motion and

$$\sinh(x) = \frac{e^x - e^{-x}}{2},\ \cosh(x) = \frac{e^x + e^{-x}}{2},$$
$$\frac{d}{dx}\sinh(x) = \cosh(x),\ \frac{d}{dx}\cosh(x) = \sinh(x),\ \cosh(x)^2 - \sinh(X)^2 = 1$$

Let also $B(t)$ and $\beta(t)$ independent Brownian motions, and

$$X(t) = e^{B(t)} \int_0^t e^{-B(s)} d\beta(s) \tag{1}$$

Apply Ito's formula to show that
the processes $(X(t) : t \geq 0)$ and $(\sinh(W(t)) : t \geq 0)$ have the same distributions,
i.e. they satisfy the same stochastic differential equation in Ito sense, possibly driven by different Brownian motions.

4. (*Brownian bridge and initially enlarged Brownian filtration) In this exercise we show that from the point of view of someone knowing the final value $W(T)$ of a Brownian motion $\big(W(t)\big)_{t\in[0,T]}$, the Brownian motion is not a martingale anymore in the bigger filtration and it becomes a Brownian bridge. If you are interested you may look at a couple of books written by Thierry Jeulin and Marc Yor on the theory of enlargement of filtrations.*

   Let $\big(W(t) : t \in [0,T]\big)$ be a Brownian motion.

   (a) *Prove that*

   $$\left(\int_0^T \frac{W(T)-W(s)}{T-s}ds\right) \in L^1(\mathbf{P})$$

   Hint: use Fubini together with the scaling property of Brownian motion.

   (b) *For $0 \le t < T$, compute the conditional density $p_{T,t}(y|x)$ of $W(T)$ conditionally on $W(t)$, such that for every bounded Borel measurable test function f*

   $$M(t,f) = \mathbb{E}\big(f(W(T))\big|\mathcal{F}_t^W\big) = \mathbb{E}\big(f(W(T))\big|W(t)\big) = \int_{\mathbb{R}} f(y)p_{T,t}(y|W(t))dy$$

   Hint. *Note that since $(W(t), W(T))$ are jointly Gaussian, the conditional density of $W(T)$ given $W(t)$ has to be Gaussian ...*

   (c) *Show that $\forall y \in \mathbb{R}$*

   $$p_{T,t}(y|W(t)) \text{ for } 0 \le t < T$$

   is a non-negative martingale in the open interval $[0,T)$, and compute its representation as a stochastic exponential.

   (d) *Show that however*

   $$p_{T,t}(y|W(t))$$

   is not a martingale in the closed interval $[0,T]$! (the conditional probability converges in distribution to point mass (a Dirac δ-function) as $(t-T) \downarrow 0$.

   (e) *Compute the predictable covariations*

   $$\langle p_{\cdot,T}(y|W(\cdot)), W\rangle_t \text{ and } \langle M(\cdot,f), W\rangle_t$$

   (f) *Show that $0 \le s \le t < T$, $A \in \mathcal{F}_s$ and bounded Borel function $f(y)$,*

   $$\mathbb{E}\bigg(\big(W(t)-W(s)\big)f(W(T))\mathbf{1}_A\bigg) = \mathbb{E}\bigg(f(W(T))\mathbf{1}_A\int_s^t \frac{W(T)-W(u)}{T-u}du\bigg)$$

   Hint. *take the conditional expectation of $f(W_T)$ w.r.t. $\mathcal{F}_t$ and rewrite it as integral w.r.t. the conditional density of $W(T)$ depending on the value $W(t)$ at time $t \le T$, use then the properties of the predictable covariation computed above.*

(g) *Show that this is equivalent to*

$$B(t) := W(t) - \int_0^T \frac{W(T) - W(u)}{T - u} du \tag{2}$$

being a martingale w.r.t. $\mathbf{P}$ *in the initially enlarged filtration* $\mathbb{G} = (\mathcal{G}_t : 0 \leq t \leq T)$ *with* $\mathcal{G}_t = \sigma\big(W(T), W(s) : 0 \leq s \leq t\big)$. *Note also that*

$$W(t) = B(t) + \int_0^T \frac{W(T) - W(u)}{T - u} du$$

Hint: show that for the enlarged σ*-algebrae we have*

$$\mathcal{G}_s := \mathcal{F}_s \vee \sigma(W(T)) = \sigma\big(A \cap \{W(T) \in B\} : \ A \in \mathcal{F}_s, B \in \mathcal{B}(\mathbb{R})\big)$$

(h) *Show that* $B(t)$ *is a Brownian motion independent from the random variable* $W(T)$ *in the initially enlarged filtration* $\mathbb{G}$.

Hint: *you can use the Lévy characterization of Brownian motion, as a continuous martingale with predictable variation equal to* t.

(i) *Let* $B(t)$ *be a Brownian motion. For each* $y \in \mathbb{R}$ *the solution of the linear SDE*

$$X(t) = B(t) + \int_0^t \frac{y - X(s)}{T - s} ds, \quad 0 \leq t \leq T$$

is called Brownian bridge pinned to y *at time* T, *which coincides in law with the distribution of the Brownian motion* $(W(t) : 0 \leq t \leq T)$ *conditioned on the event* $\{W(T) = y\}$.

(j) *Solve the linear SDE for* $X(t)$,

Hint*: look at eq.*(2).

(k) *show that* $X(t)$ *is Gaussian,*

(l) *compute* $\mathbb{E}(X(t))$ *(depending on* y*)*

(m) *compute Covariance*$(X(t), X(s))$ *for* $0 \leq s, t \leq T$.

(n) *Show that* $X(T) = y$ *with* $\mathbf{P} = 1$.

5. *Conformal martingales and conformal invariance (this topic is discussed in Revuz-Yor book and also in Morters and Peres book Brownian motion).*

Consider a complex-valued continuous local martingale

$$Z_t = X_t + iY_t$$

where X_t *and* Y_t *are* $\mathbb{R}$*-valued continuous local martingales in a filtration* $\mathbb{F}$.

(a) *Prove that there is an unique continuous* $\mathbb{C}$*-valued process of finite variation* $\langle Z, Z \rangle_t$ *such that*

$$Z(t)^2 - \langle Z, Z \rangle_t$$

is a complex local martingale.

(b) *Prove that the following statements are equivalent:*

i. $Z(t)^2$ *is a* $\mathbb{F}$*-local martingale.*

ii. $\langle Z, Z\rangle_t = 0\ \forall t$

iii. $\langle X, X\rangle = \langle Y, Y\rangle$ *and* $\langle X, Y\rangle = 0$

Such $\mathbb{C}$*-valued local martingales are called conformal local martingales. For example, if* $B(t)$ *and* $\beta(t)$ *are independent* $\mathbb{R}$*-valued Brownian motions in the filtration* $\mathbb{F}$*, the* $\mathbb{C}$*-valued planar Brownian motion*

$$W(t) = B(t) + i\beta(t) \tag{3}$$

is a conformal martingale.

(c) *Use Dambis-Dubis-Schwartz random-time-change theorem to show that if* $Z(t) = X(t) + iY(t)$ *is a* $\mathbb{C}$*-valued continuous conformal local martingale in a filtration* $\mathbb{F}$*, there exists a* $\mathbb{C}$*-valued planar Brownian motion, such that*

$$Z(t) = W(\langle X, X\rangle_t)$$

with

$$W(u) = X(\tau(u)) + iY(\tau(u)), \quad \tau(u) = \inf\big\{t : \langle X, X\rangle_t \geq u\big\}$$

(d) *Show that for a conformal local martingale* $Z(t) = X(t) + iY(t)$

$$\langle X, X\rangle_t = \langle \Re(Z), \Re(Z)\rangle_t = \frac{1}{2}\langle Z, \overline{Z}\rangle_t \tag{4}$$

(e) *Let* $H(t)$ *a* $\mathbb{C}$*-valued bounded progressive process, and* $Z(t)$ *a conformal local martingale. Show that*

$$U(t) = \int_0^t H(s)dZ(s) =$$

$$\int_0^t \Re H(s)dX(s) - \int_0^t \Im H(s)dY(s) + i\bigg(\int_0^t \Im H(s)dX(s) + \int_0^t \Re H(s)dY(s)\bigg)$$

is a conformal martingale with

$$\langle U, \overline{U}\rangle_t = \int_0^t |H(s)|^2 d\langle Z, \overline{Z}\rangle_s$$

(f) *Let* $Z(t)$ *a continuous conformal local martingale and* $F : \mathbb{C} \to \mathbb{C}$ *twice differentiable as function of two real variables. Use Ito formula to show*

$$F(Z(t)) = F(Z(0)) + \int_0^t \frac{\partial}{\partial z}F(Z(s))dZ_s + \int_0^t \frac{\partial}{\partial \overline{z}}F(Z(s))d\overline{Z}(s) + \frac{1}{4}\int_0^t \Delta F(Z(s))d\langle Z, \overline{Z}\rangle_s$$

where for $z = x + iy$

$$\Delta F(z) = \Delta(F(x,y)) = \frac{\partial^2}{\partial x^2}F(x,y) + \frac{\partial^2}{\partial y^2}F(x,y)$$

$$= \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)F(x,y) = \frac{\partial^2}{\partial z \partial \overline{z}}F(x,y)$$

is the Laplacian.

(g) *We say that F is harmonic if $\Delta F(z) = 0\ \forall z \in \mathbb{C}$. Show that if F is harmonic and $Z(t)$ a continuous conformal local martingale, then $F(Z(t))$ is a local martingale.*

(h) *Let*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

A function $F : \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if

$$\frac{\partial}{\partial \bar{z}} F(z) = 0$$

and in such case we set

$$F'(x) = \frac{\partial}{\partial z} F(z)\ .$$

If $F(z)$ is holomorphic it is necessarily harmonic. Show that in such case, if $Z(t)$ a continuous conformal local martingale, then $F(Z(t))$ is a continuous conformal local martingale with

$$F(Z(t)) = F(Z(0)) + \int_0^t F'(Z(s))dZ(s)$$

(i) *Show that if $W(t)$ is a $\mathbb{C}$-valued planar Brownian motion and $F : \mathbb{C} \to \mathbb{C}$ is holomorphic and non-constant*

$$F(W(t)) = F(W(0)) + \widetilde{W}(\langle X, X\rangle_t)$$

where $X(t) = \Re F(W_t)$ and

$$\langle X, X\rangle_t = \int_0^t |F'(W(s))|^2 ds$$

and $\widetilde{W}(t)$ is a $\mathbb{C}$-valued planar Brownian motion.
In other words $F(W(t))$ is a time changed Brownian motion.
On the complex plane, the paths of $W(t)$ and $F(W(t))$ have the same law. We say that Brownian motion is **conformally invariant** *.*