

Stochastic analysis, spring 2018, Home Exam

Solve three out of these four exercises. You can collaborate with fellow students, and if necessary the teacher for help.

To be returned by September 1st 2018.

1. (Change of measure for counting process).

We work on the probability space (Ω, \mathcal{F}, P) equipped with a filtration $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$. Consider a counting process $N(t)$ with values in \mathbb{N} which means $P(N(t) < \infty) = 1 \forall t$, adapted to the filtration \mathbb{F} , (that is a piecewise constant process with jumps $\Delta N(t) = (N(t) - N(t-)) \in \{0, 1\}$, let $\Lambda(t) = N^p(t)$ dual predictable projection w.r.t (P, \mathbb{F}) (also called compensator), and let $\bar{N}(t) = N(t) - \Lambda(t)$ be the compensated process which is a locally bounded local martingale, (since it has bounded jumps).

We assume that the jumps times of $N(t)$ are totally inaccessible, which means that the compensator has continuous paths.

- (a) What is the (\mathbb{F}, P) -predictable variation $\langle \bar{N}, \bar{N} \rangle_t$ of $\bar{N}(t)$?

Hint: recall that the predictable variation is the compensator of the quadratic variation $[\bar{N}, \bar{N}]_t$.

- (b) For a non-negative \mathbb{F} -predictable process $\rho(t)$, let

$$Z(t, \omega) = \exp\left(\int_0^t \log(\rho(s))N(ds) - \int_0^t (\rho(s) - 1)\Lambda(ds)\right) = \exp\left(-\int_0^t (\rho(s) - 1)\Lambda(ds)\right) \prod_{s \leq t} \rho(s)^{\Delta N(s)} \geq 0$$

Show that $Z(t)$ solves pathwise the linear differential equation

$$Z(t) = 1 + \int_0^t Z(s-)(\rho(s) - 1)\bar{N}(ds)$$

and with the stochastic exponential notation

$$Z(t) = \mathcal{E}((\rho - 1) \cdot \bar{N}).$$

- (c) Show that $Z(t)$ is a (P, \mathbb{F}) -locally square integrable martingale, and it is also a supermartingale.

Hint: $Z(t) \geq 0$ and you can use Fatou lemma for the localized process.

- (d) Show that $Z(t)$ is a true martingale if and only if $E_P(Z(t)) = 1 \forall t$
- (e) Show that the P -almost sure limit $Z(\infty) = \lim_{t \rightarrow \infty} Z(t)$ exists and $Z(\infty) \in L^1(P)$ and that $Z(t)$ is a uniformly integrable martingale if and only if $E_P(Z(\infty)) = 1$
- (f) Compute the (P, \mathbb{F}) -predictable variation $\langle Z, Z \rangle_t$ and predictable co-variation $\langle Z, \bar{N} \rangle_t$.
- (g) Let $\tau_n(\omega) \uparrow \infty$ be a localizing sequence such that $(\bar{N}_{t \wedge \tau_n})_{t \geq 0}$ and $(Z_{t \wedge \tau_n})_{t \geq 0}$ are true (P, \mathbb{F}) -martingales.
For each n , consider the probability space $(\Omega, \mathcal{F}_{\tau_n})$ where τ_n is the stopped σ -algebra

Show that

$$\tilde{P}_t^{(n)}(d\omega) = Z(\tau_n \wedge t, \omega)P(d\omega)$$

is a probability measure on $(\Omega, \mathcal{F}_{t \wedge \tau_n})$ and when $0 \leq s \leq t$, $m \leq n$
 $\tilde{P}_t^{(n)} = \tilde{P}_s^{(m)}$ on $\mathcal{F}_{s \wedge \tau_m}$.

- (h) Show that there exists a probability measure \tilde{P} on $(\Omega, \mathcal{F}_\infty)$ such that
 $\tilde{P} = \tilde{P}_t^{(n)}$ on $\mathcal{F}_{t \wedge \tau_n}$.
and

$$\frac{d\tilde{P}|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}}(\omega) = Z(t, \omega)$$

- (i) Show that the nondecreasing process

$$\tilde{\Lambda}(t) = \int_0^t \rho(s)\Lambda(ds)$$

is the (\mathbb{F}, \tilde{P}) -dual predictable projection of N under the new measure,
equivalently $\tilde{N}(t) = N(t) - \tilde{\Lambda}(t)$ is a \mathbb{F} -local martingale with respect
to \tilde{P} .

Hint: it is enough to show that the product $(Z(t)\tilde{N}(t))$ is a (P, \mathbb{F}) -
local martingale.

- (j) Let $N(t)$ be a Poisson process with intensity 1, which means $N^p(t) =$
 t is the compensator with respect to P and for some $\rho > 0$ let

$$Z(t) = \rho^{N(t)} \exp((1 - \rho)t)$$

Show that $N(t)$ is a Poisson process with intensity ρ under the prob-
ability Q with $\frac{dQ_t}{dP_t}(\omega) = Z(t, \omega)$.

2. Let $(B_t^{(1)}, \dots, B_t^{(n)} : t \geq 0)$ continuous local martingales in the filtration
 \mathbb{F} with

$$\begin{aligned} \langle B^{(i)}, B^{(i)} \rangle_t &= t, \\ \langle B^{(i)}, B^{(j)} \rangle_t &= E_P(B_t^{(i)} B_t^{(j)}) = c_{ij}t, \text{ for } i \neq j, . \end{aligned}$$

with $c_{ij} \in [-1, 1]$ constant.

- (a) Each $B_t^{(i)}$ is a Brownian motion. Why ?
(b) Use inductively Ito formula and Fubini Theorem to compute the joint
moment at time t :

$$E_P(B_t^{(1)} \dots B_t^{(n)}) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ t^{n/2} \sum_{\text{pairings}} \prod_{\text{pairs}\{i,j\}} c_{ij} & \text{if } n \text{ is even} \end{cases}$$

where when n is even, the sum is over all pairings of $1, \dots, n$ into $n/2$
pairs, where the pairs are disjoint and the elements of the pairs are

distance. For each pairing we then take the product over the pairs of the pairing.

This is Wick's formula for zero mean jointy Gaussian random variables (in the literature usually the proof is based on the moment generating function).

Hints: Compute the semimartingale decomposition of the product $B_t^{(1)} \dots B_t^{(n)}$, and show that the local martingale is a true martingale (which therefore has zero expectation).

Hint: Let's see how this works in practice, for example

$$E_P(B_t^{(1)} B_t^{(2)} B_t^{(3)} B_t^{(4)}) = (c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23})t^2$$

since we can form disjoint pairs in three possible way, and each pairing contributes with the product of two covariances.

Another example would be

$$E_P((B_t^{(1)})^2 B_t^{(2)} B_t^{(3)}) = (c_{11}c_{23} + 2c_{12}c_{13})t^2$$

and

$$E_P((B_t^{(1)})^2 (B_t^{(2)})^2) = (c_{11}c_{22} + 2c_{12}^2)t^2$$

This can be proved by using Ito formula to write the semimartingale decomposition of the product, and then arguing that the martingale part has zero mean.

3. Let (B_t) be a standard Brownian motion, denote $i = \sqrt{-1}$ as usual. Recall that

$$\begin{aligned} Z(t, \theta) &= \exp\left(i\theta B_t + \frac{1}{2}\theta^2 t\right) = \\ &= \cos(\theta B_t) \exp(\theta^2 t/2) + i \sin(\theta B_t) \exp(\theta^2 t/2) = M_t(\theta) + iN_t(\theta) \end{aligned}$$

is a complex valued martingale $\forall \theta \in \mathbb{R}$, that is both real and imaginary parts are martingales.

- (a) Compute the brackets $\langle M(\theta), M(\theta) \rangle_t, \langle N(\varphi), N(\varphi) \rangle_t, \langle M(\theta), N(\varphi) \rangle_t$.

Note that for different angles θ, φ you get different martingales,

$M_t(\theta) + iN_t(\theta)$ and $M_t(\varphi) + iN_t(\varphi)$.

the question is to compute the quadratic cross-covariation also when θ and φ are different.

- (b) Compute the Ito-Clarck martingale representation of the square integrable random variable $X(T)$ below with respect to the the Brownian motion $B(t)$

$$X(T) = \sin(\theta B(T)) \cos(\varphi B(T)) = E(\sin(\theta B(T)) \cos(\varphi B(T))) + \int_0^T Y(s) dB(s)$$

i.e. compute the expectation and find the adapted integrand process $Y(s)$.

Hints: rewrite

$$X_T = cM_T(\theta)N_T(\varphi)$$

with $c = \exp(-(\theta^2 + \varphi^2)T/2)$, and use integration by parts, to find the martingale decomposition of the product $(M_t(\theta)N_t(\varphi))$.

4. Let $X_T = \exp(\theta B_T)B_T^2$, where $\theta \in \mathbb{R}$.

(a) Show that $X_T \in L^2(\Omega)$.

Hint: Note that $E(\exp(\theta B_T)) = \exp(\theta^2 T/2) < \infty \forall \theta \in \mathbb{R}$.

Note also that the exponential function grows faster than any polynomial, in particular for all $\varepsilon > 0 \exists C_\varepsilon > 0$ such that

$$x^2 \leq C_\varepsilon (\exp(\varepsilon x) + \exp(-\varepsilon x)) \quad \forall x$$

Using this it is easy to show a.

(b) Compute $E_P(X_T)$.

Hint: the idea is that $\exp(\theta B_T - \theta^2/2)$ corresponds to a change of measure from the measure P to the measure P_θ with likelihood ratio

$$\frac{dP_\theta}{dP}(\omega) = \exp(\theta B_T - \theta^2/2)$$

Under the new measure P_θ , B_T has Gaussian distribution with mean θT and variance T . This follows simply by writing the product of exponentials as exponential of sum and completing the squares, inside the integral

$$\frac{1}{\sqrt{2\pi T}} \int_{\mathbb{R}} \exp(-\frac{x^2}{2T}) \exp(\theta x) x^2 dx$$

(c) Compute the Ito-Clarck martingale representation of X_T .

Hint: one approach is to compute for $f(x) = \exp(\theta x)x^2$ the conditional expectation at $t \in [0, T]$

$$E_P(f(B_T)|\mathcal{F}_t^B) = E_P(f(B_T)|B_t) = \int_{\mathbb{R}} f(x) \phi\left(\frac{x - B_t}{\sqrt{T-t}}\right) \frac{1}{\sqrt{T-t}} dx$$

where $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ is the standard Gaussian density. Use Ito formula to compute the Ito representation of the conditional density of B_T conditionally on B_t ,

$$\phi\left(\frac{x - B_t}{\sqrt{T-t}}\right) \frac{1}{\sqrt{T-t}}$$

which is a martingale $\forall y \in \mathbb{R}$, and switch the order of Ito and Lebesgue integrals. Recall the Gaussian integration by parts formula $\phi'(x) = -x\phi(x)$.