Stochastic analysis I MAST31706 Home Exam, Fall Semester 2022

To be returned by 17.12.2022. You can use the course materials and any book or paper you may find relevant, you can also collaborate with fellow students, and ask the teacher for explanations and hints.

1. Consider a two-dimensional pure-jump Lévy process $X(t) = (X_1(t), X_2(t))$, with Lévy measure on $\mathbb{R}^+ \times \mathbb{R}$

$$\ell(dx_1, dx_2) = \alpha \exp\left(-\beta x_1 - \frac{x_2^2}{2x_1}\right) x_1^{-3/2} \mathbf{1}(x_1 > 0) dx_1 dx_2$$
$$= \alpha\sqrt{2\pi} \exp\big(-\beta x_1) x_1^{-1} \mathbf{1}(x_1 > 0) dx_1 \times \exp\big(-\frac{x_2^2}{2x_1}\big) \sqrt{\frac{1}{2\pi x_1}} dx_2$$

with $\alpha, \beta > 0$, where you recognize the Lévy measure of the Gamma process for the first component times a conditional zero mean Gaussian distribution for the second component.

In fact $X_1(t)$ as a non-decreasing Lévy process is a subordinator, while $X_2(t) = W(X_1(t))$ is a subordinated Brownian motion, i.e. obtained by taking an independent Brownian motion W sampled under the random time change $t \mapsto X_1(t)$.

   (a) Use the Lévy Khinchine formula to compute the characteristic function of $X(t)$

$$\mathbb{E}\big(\exp(i\theta \cdot X(t))\big), \quad \theta \in \mathbb{R}^2$$

   Hint the Frullani integral formula can be useful.

   (b) Compute the Laplace transform of $X_1(t)$ the Fourier transform of $X_2(t)$, and the Fourier transform of $(X_1(t) + X_2(t))$.

   (c) Compute $\mathbb{E}(X_k(t))$, $k = 1, 2$.

   (d) Show that $\mathbf{P}$-almost surely $X(t)$ has finite total variation on finite intervals, equivalently both components $X_1(t)$ and $X_2(t)$ have finite total variation on finite intervals.

   (e) Compute the expectation of the total variation of X_k in an interval $[0, t]$, $k = 1, 2$.

   (f) Compute the expectation of the quadratic covariation matrix $\mathbb{E}\big([X_k, X_\ell]_t\big)$ $1 \leq k, \ell \leq 2$, where the quadratic covaration is given by

$$[X_k, X_\ell]_t = \sum_{s \leq t} \Delta X_k(s) \Delta X_\ell(s)$$

   (g) Compute joint quadratic moments

$$\mathbb{E}\big(X_k(t) X_\ell(s)\big) \quad t \geq s \geq 0, \quad 1 \leq k, \ell \leq 2.$$

   (h) Compute the $4th$-cumulants of the random variables $X_1(t)$, $X_2(t)$ and $(X_1(t) + X_2(t))$.
   Hint When it exists, the n-th cumulant of a random variable Y is given by

$$\kappa_n(Y) = (-i)^n \frac{\partial^n}{\partial \theta^n} \log \mathbb{E}\big(\exp(i\theta Y)\big)\Big|_{\theta=0}.$$

(i) Compute the 4th moments

i. $\mathbb{E}(X_1(t)^4)$,

ii. $\mathbb{E}(X_2(t)^4)$,

iii. $\mathbb{E}\big((X_1(t)+X_2(t))^4\big)$.
Hint: one way to do this, (not the only way) is to condition on the subordinator process.

2. (a) Consider a Borel space $(\mathbf{T}, \mathbf{B})$. $\mathcal{B}^{\otimes n}$ denotes the n-fold product σ-algebra. Let

$$\Delta_n = \{t = (t_1, \ldots, t_n) \in \mathbf{T}^n : \exists j < k \text{ with } \quad t_j = t_k\}$$

be the diagonal set in the product space $\mathbf{T}^n$.
Prove that $\Delta_n \in \mathcal{B}^{\otimes n}$.

(b) From now on we assume that the measure ν is σ-finite and **non-atomic**, $\nu(\{t\}) = 0 \ \forall t \in \mathbf{T}$.

Let $N(dt, \omega)$ be a Poisson process on the unit interval $\mathbf{T} = [0, 1]$ driven by the Lebesgue measure $\nu(dt) = dt$.
For which $\alpha \in \mathbb{R}$

$$\int_0^1 \int_0^1 |t-s|^\alpha N(dt)N(ds) < \infty \quad \mathbf{P}\text{-almost surely ?}$$

(c) For such α compute also the expectation

$$\mathbb{E}\bigg(\int_0^1 \int_0^1 |t-s|^\alpha N(dt)N(ds)\bigg)$$

(d) For which $\beta \in \mathbb{R}$

$$\int_{[0,1]^2 \setminus \Delta_2} |t-s|^\beta N(dt)N(ds) < \infty \quad \mathbf{P}\text{-almost surely ?}$$

(e) For such β, compute the expectation

$$\mathbb{E}\bigg(\int_{[0,1]^2 \setminus \Delta_2} |t-s|^\beta N(dt)N(ds)\bigg)$$

(f) Show that under square integrability assumptions,

$$\int_{[0,1]^2 \setminus \Delta_2} f(t_1,t_2)\bar{N}(dt_1)\bar{N}(dt_2) = \int_{[0,1]^2 \setminus \Delta_2} \widetilde{f}(t_1,t_2)\bar{N}(dt_1)\bar{N}(dt_2) = \delta^2(\widetilde{f}) \tag{1}$$

where

$$\widetilde{f}(t_1, \ldots, t_n) = \frac{1}{n!} \sum_{\text{permutations } \pi} f(t_{\pi(1)}, \ldots, t_{\pi(n)})$$

is the *symmetrization* of f, and $\delta^2(\widetilde{f})$ is an iterated Skorokhod integral, where first we integrate the time parameter t_1, obtaining

$$\delta(\widetilde{f}(\cdot, t_2))$$

which is random process indexed by t_2, and then we take a second Skorokhod integral in the time parameter t_2.

(g) Compute the square norm $\mathbb{E}(\delta^2(\widetilde{f})\delta^2(\widetilde{f}))$ of the iterated Skorokhod integral in terms of the $L^2(T^2, \nu^{\otimes 2})$ norm of $\widetilde{f}$.

(h) For which $\gamma \in \mathbb{R}$ the iterated L^2 integral below with respect to the compensated Poisson process $\bar{N}(B, \omega) = N(B, \omega) - |B|$ with N driven by the Lebesgue measure is well defined ?

$$\int_{[0,1]^2 \setminus \Delta_2} |t - s|^\gamma \bar{N}(dt)\bar{N}(ds)$$

(i) For such γ, compute the squared L^2-norm

$$\mathbb{E}\left(\left\{ \int_{[0,1]^2 \setminus \Delta^2} |t - s|^\gamma \bar{N}(dt)\bar{N}(ds) \right\}^2 \right)$$

(j) Show that for every Borel set $B \subseteq [0, 1]$,

$$c_k\big(N(B), \nu(B)\big) = \frac{1}{k!}\delta^k\big(\mathbf{1}_B^{\otimes k}\big) = \frac{1}{k!}\int_{[0,1]^k \setminus \Delta_k} \mathbf{1}_B(t_1) \dots \mathbf{1}_B(t_k)\bar{N}(dt_1) \dots \bar{N}(dt_k)$$
$$= \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} \mathbf{1}_B(t_1) \dots \mathbf{1}_B(t_k)\bar{N}(dt_1) \dots \bar{N}(dt_{k-1})\bar{N}(dt_k)$$

where $f \otimes g(t_1, t_2) = f(t_1)g(t_2)$ denotes the tensor product of functions and $f^{\otimes 2} = f \otimes f$, and $c_k(N, \nu)$ are the unnormalized Charlier polynomials introduced in the first problem sheet of the course.