Stochastic analysis I MAST31706 Home Exam, Fall Semester 2023

To be returned by 22.12.2023. You can use the course materials and any book or paper you may find relevant, you can also collaborate with fellow students, and ask the teacher for explanations and hints.

Let's agree that it is sufficient to answer correctly to 50% of the questions and subquestions to pass the exam with best grades. Choose your 50% of questions as you like.

Join the students telegram channel for this course at
`https://t.me/+uesNRjzKrz82OWI0`

1. Let $N(B)$ a simple point process (p.p.) on a Borel space $(T, \mathcal{B})$, not necessarily a Poisson p.p.

   For measurable $f : T \mapsto \mathbb{R}_+$ denote the Laplace transform functional of the p.p. N as

   $$L_N(f) = \mathbb{E}\Big[\exp\Big(-\int_T f(t)N(dt)\Big)\Big]$$

   Prove that $\forall B \in \mathcal{B}$

   $$\mathbf{P}\big(N(B) = 0\big) = \lim_{t\uparrow\infty} L_N\big(t\mathbf{1}_B\big)$$

   where $\mathbf{1}_B(t)$ is the indicator function of B.

2. Let $N(B)$ a point process (p.p.) on a Borel space $(T, \mathcal{B})$. Show that N is a Poisson p.p. with a driving measure ν if and only if $\forall B \in \mathcal{B}$ $N(B)$ is a Poisson random variable of mean parameter $\nu(B)$ (The surprising fact is that independence of N over disjoint subsets follows from this).

3. Let $N_1(B)$ and $N_2(B)$ point process on a Borel space $(T, \mathcal{B})$, not necessarily Poisson processes. For $B \in \mathcal{B}$ let $\zeta_k(B) = \mathbf{P}\big(N_k(B) = 0\big)$ $k = 1, 2$ be the respective avoidance probabilities functions of N_1 and N_2.

   Show that one can construct a point process N with avoidance probability function $\zeta_1(B) \times \zeta_2(B)$

4. Let $N(B)$ a point process on a Borel space $(T, \mathcal{B})$ with avoidance probability function $\zeta(B) = \mathbf{P}(N(B) = 0)$, $B \in \mathcal{B}$, which is not necessarily a Poisson p.p., and let $\phi_X(\theta) = \mathbb{E}\big[\theta^X\big]$ be the moment generating function of an $\mathbb{N}$-valued random variable X. Show that one can construct a point process with avoidance probability function $\phi_X(\zeta(B))$, $B \in \mathcal{B}$.

5. A Cox point process on the Borel space $(B, \mathcal{B})$, also known as double stochastic Poisson process, is a point process $C(\omega, B)$ with driving random measure $\mu(\omega, B) \geq 0$, which is $\mathbf{P}$-a.s. non-atomic and σ-finite, such that conditionally on the random measure μ, C is a Poisson process driven by μ. Equivalently, $\forall k$, $B_0, B_1 \ldots, B_k \in \mathcal{B}$ and bounded measurable test functions $f(x_1, \ldots, x_k)$

   $$\mathbb{E}\big[f(\mu(B_1), \ldots, \mu(B_k))\mathbf{1}(C(B_0) = 0)\big] = \mathbb{E}\big[f(\mu(B_1), \ldots, \mu(B_k))\exp(-\mu(B_0))\big]$$

   Is the cluster point process discussed in Ex.3 of the Problem sheet 3 a Cox process ? If so, what is the random measure $\mu(\omega, dt)$ driving the Cox process for that cluster point process ? Explain your answer.

6. Consider a Cox point process $C(B)$ on a Borel space $(T, \mathcal{B})$ with random intensity $\theta(\omega)\nu(dt)$ where $\theta(\omega) \geq 0$ is a random variable with $\mathbb{E}(\theta) = \bar{\theta} \in (0, \infty)$ and $\nu(dt)$ is a deterministic σ-finite non-atomic measure. Without loss of generality you can assume that $\bar{\theta} = 1$ (why ?).

   (a) Compute the Laplace transform functional and the avoidance probabilities of C.

   (b) Compute the Palm measure of the Cox process C at a generic point $t \in T$.

   (c) Let $N(B)$ be a Poisson point process driven by the mean intensity measure $\nu(B) = \mathbb{E}(C(B))$ of the Cox process C. Show that

$$\mathbf{P}(N(B) = 0) \leq \mathbf{P}(C(B) = 0) \quad \forall B \in \mathcal{B}$$

7. Let $(T, \mathcal{B})$ a Borel space equipped with a **finite** non-atomic positive measure $\alpha(dt)$, $\alpha(T) < \infty$. Let $N(dt, dx)$ be a Poisson point process on the product space $\mathbb{R}_+ \times T$ with driving measure

$$\nu(dx, dt) = x^{-1}e^{-x}dx\ \alpha(dt), \quad x > 0$$

   We define the Gamma random measure on $(T, \mathcal{B})$ as the following random integral w.r.t. the Poisson p.p. N

$$\mu(B) = \int_T \int_0^\infty \mathbf{1}_B(t) x N(dx, dt)$$

   (a) Compute the Laplace transform functional

$$L_\mu(f) = \mathbb{E}\left[\exp\left(-\int_T f(t)\mu(dt)\right)\right]$$

   for measurable test functions $f(t) \geq 0$.

   (b) Show that $\forall B \in \mathcal{B}$ the random variable $\mu(B)$ has Gamma distribution and compute its parameters.

   (c) Compute the mean intensity measure $B \mapsto \mathbb{E}\big(\mu(B)\big)$

   (d) Compute the Palm measure of the random measure μ at point $t \in T$.

   (e) By normalizing the random measure μ, we obtain the Dirichlet random probability measure

$$\Pi(B) = \mu(B)/\mu(T), \quad B \in \mathcal{B}$$

   (f) Prove that the Dirichlet random probability measure $(\Pi(B) : B \in \mathcal{B})$ is independent from the random variable $\mu(T)$.

   Hint Equivalently, show that $\forall n$, and $B_1, \ldots, B_n \in \mathcal{B}$ disjoint with $\bigcup_{k=1}^n B_k = T$ show that $(\Pi_1(B_1), \ldots, \Pi(B_n))$ is a random probability vector independent from $\mu(T) = \mu(B_1) + \cdots + \mu(B_n)$. This can be done by changing variables and computing the conditional density. The distribution of $(\Pi_1(B_1), \ldots, \Pi(B_n))$ obtained in this way is called Dirichlet distribution.

(g) Compute the Palm measure of the Dirichlet random probability measure at a point $t \in T$?

(h) Prove the Cifarelli-Regazzini identity

$$\mathbb{E}\left[\left(1+\int_T f(t)\Pi(dt)\right)^{-\alpha(T)}\right] = L_\mu(f)$$

Hint: $\mu(B) = \mu(T)\Pi(B)$, use the characteristic function of the Gamma r.v. $\mu(T)$ together with conditioning under independence.

8. Let N be an homogeneous Poisson p.p. on $\mathbb{R}_+$ with driving measure $\lambda\, dt$ proportional to Lebesgue measure.

Denote the counting process as $N(t) = N((0,t])$, and the compensated counting process $\bar{N}(t) = N(t) - \lambda t$, and consider the running maximum or current record process

$$\bar{N}^*(t) = \max_{0<s\leq t} \bar{N}(s)$$

Hint it is useful to draw a graph with the trajectories of $\bar{N}(t)$ and $\bar{N}^*(t)$.

(a) Show that

$$M(t) := \bar{N}^*(t) - \int_0^t (\bar{N}(s-) + 1 - \bar{N}^*(s-))^+ \lambda ds$$

is a martingale in the filtration $\mathbb{F}^N$ generated by N. We used the notation $x^+ = \max\{x, 0\}$.

(b) For fixed $T > 0$ Compute the martingale representation of the random variable $\bar{N}^*(T)$.
Hint apply the Ito Clarck Ocone formula in Poisson space.

(c) Let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k$ the random record times when the running maximum changes, it can change only by positive jumps,

$$\tau_k = \inf\{t > \tau_{k-1} : \bar{N}(t) = \bar{N}(t)^*\}, \quad k > 0,$$

let $\Delta\tau_k = \tau_k - \tau_{k-1} > 0$ and

$$\Delta X_k = \bar{N}^*(\tau_k) - \bar{N}^*(\tau_k-) = \bar{N}^*(\tau_k) - \bar{N}^*(\tau_{k-1}) > 0$$

the jump times and jump sizes of the $\bar{N}^*$ jumps. It can be shown that $(\Delta\tau_k, \Delta X_k)_{k\in\mathbb{N}}$ are i.i.d. random pairs. Consider the counting process

$$C(B) = \sum_{k=1}^{\infty} \mathbf{1}(\tau_k \in B), \quad B \in \mathcal{B}(\mathbb{R}_+)$$

counting the record times of $\bar{N}$ and express

$$\frac{1}{T}\mathbb{E}\big[\bar{N}^*(T)\big]$$

by using the mean intensity of C and its Palm measure. **Hint:** apply P. Bremaud swiss-army-knife formula.

9. Let $A(t)$ be a Gamma subordinator (Lévy process with positive jumps) with Lévy measure

$$\ell(dx) = ae^{-bx}x^{-1}dx, \quad x > 0$$

and $W(t)$ independent Brownian motion, which is a process with continuous paths and independent stationary Gaussian increments, with

$$W(0) = 0, \quad \mathbb{E}(W(t) - W(s)) = \mathbb{E}(W(t-s)) = 0,$$
$$\mathbb{E}((W(t) - W(s))^2) = \mathbb{E}(W(t-s)^2) = t - s, \quad 0 \le s \le t$$

Let $X(t) = W(A(t))$ be the Gamma-subordinated Brownian motion.

(a) Show that the pair $(A(t), X(t))$ is a Lévy process and compute its Lévy measure and Laplace transform.

(b) Compute the Lévy measure and the probability marginal density of the component $X(t)$.

(c) Show that the $X(t)$ component is selfdecomposable, that is $\forall \alpha \in (0,1)$,

$$X(t) \stackrel{Law}{=} \alpha X(t) + X_\alpha(t)$$

where $X_\alpha(t)$ is an independent Lévy process and write an expression for the Lévy measure of $X_\alpha(t)$.

(d) Show that the quadratic variation of X_t

$$[X]_t = \sum_{s \le t} (\Delta X(s))^2$$

is a Lévy process and compute its Lévy measure and Laplace transform.

10. Let $N(dt, \omega)$ be a Poisson process on the unit interval $\mathbf{T} = [0,1]$ driven by the Lebesgue measure $\nu(dt) = dt$.

(a) For which $\alpha \in \mathbb{R}$

$$\int_0^1 \int_0^1 |t-s|^\alpha N(dt)N(ds) < \infty \quad \mathbf{P}\text{-almost surely ?}$$

(b) For such α compute also the expectation

$$\mathbb{E}\left(\int_0^1 \int_0^1 |t-s|^\alpha N(dt)N(ds) \right)$$

(c) For which $\beta \in \mathbb{R}$

$$\int_{[0,1]^2} |t-s|^\beta \mathbf{1}(t \neq s) N(dt)N(ds) < \infty \quad \mathbf{P}\text{-almost surely ?}$$

(d) For such β, compute the expectation

$$\mathbb{E}\Big(\int_{[0,1]^2} |t-s|^\beta \mathbf{1}(t \neq s) N(dt)N(ds)\Big)$$

(e) We denote by

$$\widetilde{f}(t_1,\dots,t_n) = \frac{1}{n!} \sum_{\text{permutations}\pi} f(t_{\pi(1)},\dots,t_{\pi(n)})$$

the *symmetrization* of a function f.
Show that under square integrability assumptions,

$\delta^2(\widetilde{f}) =$
$$\int_{[0,1]^2} \widetilde{f}(t,s)\mathbf{1}(t\neq s)\bar{N}(ds)\bar{N}(dt) = \int_{[0,1]^2} f(t,s)\mathbf{1}(t\neq s)\bar{N}(ds)\bar{N}(dt)$$

where $\bar{N} = (N-\nu)$ and $\delta^2(\widetilde{f})$ is an iterated Skorokhod integral, where first we integrate the time parameter s, obtaining

$$\delta(\widetilde{f}(\cdot,t))$$

which is random process indexed by t, and then we take a second Skorokhod integral in the time parameter t.

(f) Compute $\mathbb{E}(\delta^2(\widetilde{f})\delta^2(\widetilde{g}))$ of the iterated Skorokhod integral in terms of the $L^2(T^2,\nu^{\otimes 2})$ scalar product of $\widetilde{f},\widetilde{g}$.

(g) For which $\gamma \in \mathbb{R}$ the iterated L^2 integral below with respect to the compensated Poisson process $\bar{N}(B,\omega) = N(B,\omega) - |B|$ with N driven by the Lebesgue measure is well defined ?

$$\int_{[0,1]^2} |t-s|^\gamma \mathbf{1}(t\neq s)\bar{N}(dt)\bar{N}(ds)$$

(h) For such γ, compute the squared L^2-norm

$$\mathbb{E}\Big(\Big\{\int_{[0,1]^2} |t-s|^\gamma \mathbf{1}(t\neq s)\bar{N}(dt)\bar{N}(ds)\Big\}^2\Big)$$