Stochastic analysis I MAST31706 Home Exam, Fall Semester 2024

To be returned by 22.12.2024.

You are not asked to solve all the problems: choose as you like 4 from the problems numbered from 1 to 8 and answer to all subquestions of your chosen problems.

You can use the course materials and any book or paper you may find relevant, you can also collaborate with fellow students, and ask the teacher for explanations and hints. Welcome to join the students whatsapp group for this course using the invitation link

`https://chat.whatsapp.com/E2fKQRczUrcKV0Wd55emTt`

1. Let $N(B)$ a simple point process (p.p.) on a Borel space $(T, \mathcal{B})$, not necessarily a Poisson p.p.

   For measurable $f : T \mapsto \mathbb{R}_+$ denote the Laplace transform functional of the p.p. N as

   $$L_N(f) = \mathbb{E}\Big[\exp\Big(-\int_T f(t)N(dt)\Big)\Big]$$

   (a) Prove that $\forall B \in \mathcal{B}$

   $$\mathbf{P}\big(N(B) = 0\big) = \lim_{t\uparrow\infty} L_N\big(t\mathbf{1}_B\big)$$

   where $\mathbf{1}_B(t)$ is the indicator function of B.

   (b) Let $N(B)$ a point process (p.p.) on a Borel space $(T, \mathcal{B})$. Show that N is a Poisson p.p. with a driving measure ν if and only if $\forall B \in \mathcal{B}$ $N(B)$ is a Poisson random variable of mean parameter $\nu(B)$ (The surprising fact is that independence of N over disjoint subsets follows from this!).

   (c) Let $N_1(B)$ and $N_2(B)$ point process on a Borel space $(T, \mathcal{B})$, not necessarily Poisson processes. For $B \in \mathcal{B}$ let $\zeta_k(B) = \mathbf{P}\big(N_k(B) = 0\big)$ $k = 1, 2$ be the respective avoidance probabilities functions of N_1 and N_2.

   Show that one can construct a point process N with avoidance probability function $\zeta_1(B) \times \zeta_2(B)$

   (d) Let $N(B)$ a point process on a Borel space $(T, \mathcal{B})$ with avoidance probability function $\zeta(B) = \mathbf{P}(N(B) = 0)$, $B \in \mathcal{B}$, which is not necessarily a Poisson p.p., and let $\phi_X(\theta) = \mathbb{E}\big[\theta^X\big]$ be the moment generating function of an $\mathbb{N}$-valued random variable X. Show that one can construct a point process with avoidance probability function $\phi_X(\zeta(B))$, $B \in \mathcal{B}$.

2. A Cox point process on the Borel space $(B, \mathcal{B})$, also known as double stochastic Poisson process, is a point process $C(\omega, B)$ with driving random measure $\nu(\omega, B)$, which is $\mathbf{P}$-a.s. non-atomic and σ-finite, such that conditionally on the random measure ν, C is a Poisson process driven by ν. Equivalently, $\forall k$, $B_0, B_1 \ldots, B_k \in \mathcal{B}$, also overlapping,

   $$\mathbb{E}\big[\nu(B_1)\ldots\nu(B_k)\mathbf{1}(C(B_0) = 0)\big] = \mathbb{E}\big[\nu(B_1)\ldots\nu(B_k)\exp(-\nu(B_0))\big]$$

Consider a Cox point process $C(B)$ on a Borel space $(T, \mathcal{B})$ with random intensity $\theta(\omega)\nu(dt)$ where $\theta(\omega) \geq 0$ is a random variable with $\mathbb{E}(\theta) = \bar{\theta} = 1$ and $\nu(dt)$ is a deterministic σ-finite non-atomic measure.

(a) Compute the Palm measure of the Cox process C at a generic point $t \in T$. Compute also the Laplace transform and the avoidance probability of C.

(b) Let also $N(B)$ be a Poisson point process with driven by the mean intensity measure ν of the Cox process C.
Show that $\mathbf{P}(N(B) = 0) \leq \mathbf{P}(C(B) = 0)\ \forall B \in \mathcal{B}$.

3. Let $(T, \mathcal{B})$ a Borel space equipped with a finite non-atomic positive measure $\alpha(dt)$, $\alpha(T) < \infty$. Let $N(dt, dx)$ be a Poisson point process on the product space $\mathbb{R}_+ \times T$ with driving measure

$$\nu(dx, dt) = x^{-1}e^{-x}dx\ \alpha(dt), \quad x > 0$$

We define the Gamma random measure on $(T, \mathcal{B})$ as the following random integral w.r.t. the Poisson p.p. N

$$\mu(B) = \int_T \int_0^\infty \mathbf{1}_B(t) x N(dx, dt)$$

(a) Compute the Laplace transform functional

$$L_\mu(f) = \mathbb{E}\bigg[\exp\bigg(-\int_T f(t)\mu(dt)\bigg)\bigg]$$

for measurable test function $f(t) \geq 0$.

(b) Show that $\forall B \in \mathcal{B}$ the random variable $\mu(B)$ has Gamma distribution and compute its parameters.

(c) Show that if $\alpha(B_\ell \cap B_k) = 0\ \forall n \in \mathbb{N},\ B_1, \ldots, B_n \in \mathcal{B},\ 1 \leq k < \ell \leq n$, then the random variables $\mu(B_1) \ldots, \mu(B_n)$ are independent.

(d) Compute the mean intensity measure $\mathbb{E}\big(\mu(B)\big)$

(e) Compute the Palm measure of the random measure μ at point $t \in T$.

(f) By normalizing the random measure μ, we obtain the Dirichlet random probability measure

$$\Pi(B) = \mu(B)/\mu(T), \quad B \in \mathcal{B}$$

Prove that the Dirichlet random probability measure $(\Pi(B) : B \in \mathcal{B})$ is independent from the random variable $\mu(T)$.

Hint Equivalently, show that $\forall n$, and $B_1, \ldots, B_n \in \mathcal{B}$ disjoint with $\bigcup_{k=1}^n B_k = T$ show that $(\Pi_1(B_1), \ldots, \Pi(B_n))$ is a random probability vector independent from $\mu(T) = \mu(B_1) + \cdots + \mu(B_n)$. This can be done by changing variables and computing the conditional density. The distribution of $(\Pi_1(B_1), \ldots, \Pi(B_n))$ obtained in this way is called Dirichlet distribution.

(g) Prove the Cifarelli-Regazzini identity

$$\mathbb{E}\left[\left(1+\int_T f(t)\Pi(dt)\right)^{-\alpha(T)}\right] = L_\mu(f)$$

Hint: $\mu(B) = \mu(T)\Pi(B)$, use the characteristic function of the Gamma r.v. $\mu(T)$ together with conditioning under independence.

(h) What is the Palm probability distribution of the Dirichlet random probability measure Π at a point $t \in T$?

4. Let N be an homogeneous Poisson p.p. on $\mathbb{R}_+$ whith driving measure $\lambda\, dt$ proportional to Lebesgue measure.

Denote the counting process as $N(t) = N((0,t])$, and the compensated counting process $\bar{N}(t) = N(t) - \lambda t$, and consider the running maximum process

$$\bar{N}^*(t) = \max_{0<s\leq t} \bar{N}(s)$$

(a) Show that

$$M(t) := \bar{N}^*(t) - \int_0^t (\bar{N}(s-) + 1 - \bar{N}^*(s-))^+ \lambda ds$$

is a martingale in the filtration $\mathbb{F}^N$ generated by $N(t)$.

(b) For fixed $T > 0$ compute the martingale representation of the random variable $\bar{N}^*(T)$.

Hint apply the Ito Clarck Ocone formula for the Poisson p.p.

(c) Let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k$ the random record times when the running maximum changes (in this situation it can change only by positive jumps. :

$$\tau_k = \inf\{t > \tau_{k-1} : \bar{N}(t) = \bar{N}(t)^*\}, \quad k > 0$$

let $\Delta\tau_k = \tau_k - \tau_{k-1} > 0$ and let

$$\Delta X_k = \bar{N}^*(\tau_k) - \bar{N}^*(\tau_k-) = \bar{N}^*(\tau_k) - \bar{N}^*(\tau_{k-1}) > 0$$

the sizes of the jumps of $\bar{N}^*$. One can show that $(\Delta\tau_k, \Delta X_k)$ are i.i.d. random pairs. Consider the counting process $C(B) = \sum_{k=1}^\infty \mathbf{1}(\tau_k \in B)$ counting the record times of $\bar{N}$ and express

$$\frac{1}{T}\mathbb{E}\big[\bar{N}^*(T)\big]$$

by using the mean intensity of C and its Palm measure.

Hint apply Bremaud swiss-army-knife formula.

5. Consider a two-dimensional pure-jump Lévy process $X(t) = (X_1(t), X_2(t))$, with Lévy measure on $\mathbb{R}^+ \times \mathbb{R}$

$$\ell(dx_1, dx_2) = \alpha \exp\left(-\beta x_1 - \frac{x_2^2}{2x_1}\right) x_1^{-3/2} \mathbf{1}(x_1 > 0) dx_1 dx_2$$

$$= \alpha\sqrt{2\pi}\exp\big(-\beta x_1) x_1^{-1} \mathbf{1}(x_1 > 0) dx_1 \times \exp\big(-\frac{x_2^2}{2x_1}\big)\sqrt{\frac{1}{2\pi x_1}} dx_2$$

with $\alpha, \beta > 0$, where you recognize the Lévy measure of the Gamma process for the first component times a conditional zero mean Gaussian distribution for the second component.

In fact $X_1(t)$ as a non-decreasing Lévy process is a subordinator, while $X_2(t) = W(X_1(t))$ is a subordinated Brownian motion, i.e. obtained by taking an independent Brownian motion W with indepent time homogenous Gaussian incrementes sampled under the random time change $t \mapsto X_1(t)$.

(a) Use the Lévy Khinchine formula to compute the characteristic function of $X(t)$

$$\mathbb{E}\big(\exp(i\theta \cdot X(t))\big), \quad \theta \in \mathbb{R}^2$$

Hint Remember Frullani integral formula. Conditioning on the subordinator process could be also useful.

(b) Compute the Laplace transform of $X_1(t)$ the Fourier transform of $X_2(t)$, and the Fourier transform of $(X_1(t) + X_2(t))$.

(c) Compute $\mathbb{E}(X_k(t))$, $k = 1, 2$.

(d) Show that $\mathbf{P}$-almost surely $X(t)$ has finite total variation on finite intervals, equivalently both components $X_1(t)$ and $X_2(t)$ have finite total variation on finite intervals.

(e) Compute the expectation of the total variation of X_k in an interval $[0, t]$, $k = 1, 2$.

(f) Compute the expectation of the quadratic covariation matrix $\mathbb{E}\big([X_k, X_\ell]_t\big)$ $1 \leq k, \ell \leq 2$, where the quadratic covaration is given by

$$[X_k, X_\ell]_t = \sum_{s \leq t} \Delta X_k(s) \Delta X_\ell(s)$$

(g) Show that the $X(t)$ component is selfdecomposable, that is $\forall \alpha \in (0, 1)$,

$$X(t) \stackrel{Law}{=} \alpha X(t) + X_\alpha(t)$$

where $X_\alpha(t)$ is an independent Lévy process and write an expression for the Lévy measure of $X_\alpha(t)$.

6. (a) Let $N(dt, \omega)$ be a Poisson process on the unit interval $\mathbf{T} = [0, 1]$ driven by the Lebesgue measure $\nu(dt) = dt$.
For which $\alpha \in \mathbb{R}$

$$\int_0^1 \int_0^1 |t - s|^\alpha N(dt) N(ds) < \infty \quad \mathbf{P}\text{-almost surely ?}$$

(b) For such α compute also the expectation

$$\mathbb{E}\bigg(\int_0^1 \int_0^1 |t - s|^\alpha N(dt) N(ds)\bigg)$$

(c) For which $\beta \in \mathbb{R}$

$$\int_{[0,1]^2} |t-s|^\beta \mathbf{1}(t \neq s) N(dt)N(ds) < \infty \quad \mathbf{P}\text{-almost surely ?}$$

(d) For such β, compute the expectation

$$\mathbb{E}\bigg(\int_{[0,1]^2} |t-s|^\beta \mathbf{1}(t \neq s) N(dt)N(ds)\bigg)$$

(e) For which $\gamma \in \mathbb{R}$ the iterated L^2 integral below with respect to the compensated Poisson process $\bar{N}(B,\omega) = N(B,\omega) - |B|$ with N driven by the Lebesgue measure is well defined ?

$$\int_{[0,1]^2} |t-s|^\gamma \mathbf{1}(t \neq s) \bar{N}(dt)\bar{N}(ds)$$

(f) For such γ, compute the squared L^2-norm

$$\mathbb{E}\bigg(\bigg\{\int_{[0,1]^2} |t-s|^\gamma \mathbf{1}(t \neq s) \bar{N}(dt)\bar{N}(ds)\bigg\}^2\bigg)$$

7. Consider a Poisson point process $N(dt)$ on $\mathbb{R}_+$ with driving measure $\nu(dt)$. We denote $N(t) = N((0,t])$ and $\nu(t) = \nu((0,t])$.

   Let

$$\lambda(t,N) = \theta^{N(t-)} \exp(-\theta\nu(t))$$

   (a) Show that $\lambda(t,N)$ is predictable in the filtration $\mathbb{F}^N$ of the Poisson process and for $0 < \theta < 1$ it is uniformly bounded

$$\sup_{0<s\leq t} \| \lambda(s,N) \|_\infty < \infty \; \forall t$$

   We continue under this assumption.

   (b) Express

$$Z(\lambda,t) = \exp\bigg(\int_0^T \log\lambda(s,N)N(ds) - \int_0^t (\lambda(s,N)-1)\nu(ds)\bigg)$$

   as an Ito integral w.r.t. $\bar{N} = (N-\nu)$ and show that $Z(\lambda,t)$ is a martingale in the filtration $\mathbb{F}^N$ w.r.t. the distribution of the Poisson point process $\mathbf{P}_\nu$.

   Hint: remember the chain rule of Lebesgue Stieltjes calculus.

   (c) Show that $\mathbb{E}_{\mathbf{P}_\nu}(Z(\lambda,t)) = 1$ and

$$d\mathbf{P}_{\lambda\cdot\nu} := Z(\lambda,t) d\mathbf{P}_\nu$$

   defines a probability measure absolutely continuous w.r.t. the Poisson p.p. distribution $\mathbf{P}_\nu$ on each finite time interval $(0,t]$.

(d) Compute the relative entropies $I(\mathbf{P}_{\lambda\cdot\nu} \parallel \mathbf{P}_\nu)$ and $I(\mathbf{P}_\nu \parallel \mathbf{P}_{\lambda\cdot\nu})$ between whese two point process distributions on a finite time interval $(0,t]$.

Hint This problem with $\lambda(t)$ deterministic was discussed in Problem sheet 5.

(e) Compute the mean intensity measure of N

$$\mathbb{E}_{\lambda\cdot\nu}\big(N(B)\big) = \mathbb{E}_{\lambda\cdot\nu}\left(\frac{d\mathbf{P}_{\lambda\cdot\nu}}{d\mathbf{P}_\nu}N(B)\right) = \mathbb{E}_\nu\big(Z(\lambda,t)N(B)\big), \quad B \subseteq (0,t] \text{ Borel set}$$

under the new measure $\mathbf{P}_{\lambda\cdot\nu}$.

8. Consider a 2-dimensional Lévy process $X(t) = (Y(t), Z(t))$ ¨with Lévy measure

$$\ell(dy,dz) = \mathbf{1}(y>0)\alpha e^{-\beta y}y^{-1}dy\Big((1-e^{-y})\delta_1(dz) + e^{-x}\delta_0(dz)\Big)$$

i.e. when the Y component has a jump of size $\Delta Y(t) > 0$, with probability $(1-\exp(-\Delta X(t))$ the Z component has a simultaneous jump of size 1.

(a) Show that $Z(t)$ is a Poisson point process and compute its intensity measure.

(b) Describe the conditional law of the process $(Y(t) : t \in (0,T])$ conditionally on the process $(Z(t) : t \in (0,T])$ and compute the conditional Laplace transform of Y given Z

$$\mathbb{E}\left(\exp\left(-\int_0^t f(s)Y(ds)\right)\middle| Z(s) : 0 < s \le t\right)$$

for deterministic $f \ge 0$.