SPECTRAL THEORY (spring 2024)

**FINAL EXAM 15.5.24 at 09.15-12.15, in room B222**

No materials allowed for the test.

**NOTE: Choose only one of the exercises 4 and 5 below !**

**1.** Recall that $T \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if $\|T\|_{HS}^2 := \sum_{j=1}^{\infty} \|Te_j\|^2 < \infty$. Prove that:

(i) The infinite matrix $A = (a_{nm})_{m,n=1}^{\infty}$ is the matrix of a Hilbert-Schmidt operator $T$ if and only if $\sum_{j,k=1}^{\infty} |a_{j,k}|^2 < \infty$, and actually $\|T\|_{HS}^2 = \sum_{j,k=1}^{\infty} |a_{j,k}|^2$.

(ii) $\|T^*\|_{HS} = \|T\|_{HS}$.

(iii) The definition of the norm $\|T\|_{HS}$ is independent of the used orthonormal basis.

**2.** Let $A \in \mathcal{B}(H)$ be a bounded self-adjoint operator. Prove that

(i) $\lambda \in \sigma(A)$ if and only if $\lambda$ is in the approximative spectrum, i.e. there are vectors $x_n \in H$ with $\|x_n\| = 1$ for all $n \geq 1$

$$\lim_{n\to\infty} \|\lambda x_n - Ax_n\| = 0.$$

(ii) Either $\|A + 3I\| \geq 2$ or $\|A^2 + I\| \geq 2$.

**3.** Let $A$ be a (possibly unbounded) self-adjoint operator on $H$ such that $\sigma(A) \subset [0, \infty)$.

(i) How can you express $(A + \lambda I)^{-1}$ for $\lambda > 0$ as an integral against $d\mu^A$ (the projection valued spectral measure defined by $A$) ?

(ii) Assume that $(A+I)^{-1}$ is a compact operator on $H$. Prove or disprove: then $(A+\lambda I)^{-1}$ is a compact operator for all $\lambda > 0$.

**CHOOSE either question 4 or 5 below, not both!**

**4.** Describe the proof of the spectral theorem for a compact self-adjoint operator $T \in K(H)$ (not all details are needed, rather just the important ideas and why they might work).

**5.** Let $A \in \mathcal{B}(H)$ be a bounded self-adjoint operator. An important step of the proof given in the lectures for the spectral theorem for bounded self-adjoint operators yields the continuous spectral calculus: if $f, g$ are continuous real-valued functions on the spectrum $\sigma(A)$, we may define the operator $f(A)$ and it has (among others) the following properties:

- $f(A)$ is self-adjoint,
- $(fg)(A) = f(A)g(A)$,
- $\|f(A)\| = \sup_{\lambda\in\sigma(A)} |f(\lambda)|$,
- $\sigma(f(A)) = f(\sigma(A))$.

Describe the construction of $f(A)$ and the proofs of the above properties (not all detail are needed, rather the important ideas and why they might work).