Risk theory 14.3.2018
General examination 3h 30min
Basic calculator allowed

1. Let K_1 and K_2 be counting variables and let Q be a non-negative random variable. Assume that $\mathbb{E}(Q) = 1$. Let H be the distribution function of Q. Suppose that

$$\mathbb{P}(K_1 = k_1, K_2 = k_2) = \int_0^\infty e^{-\lambda_1 q}\frac{(\lambda_1 q)^{k_1}}{k_1!}\, e^{-\lambda_2 q}\frac{(\lambda_2 q)^{k_2}}{k_2!}\, dH(q), \quad k_1, k_2 = 0, 1, 2, \ldots$$

where λ_1 and λ_2 are positive constants. Prove that K_1 and $K_1 + K_2$ have mixed Poisson distributions.

2. The capital requirement of every insurance company in the market is determined by the normal approximation of the total claim amount of the forthcoming year. To be solvent, the initial capital together with the premium of the forthcoming year must suffice for the compensations with the probability $\geq 1 - \epsilon$. Assume that two companies in the market are both solvent. Prove that if the companies are merged then also the resulting company is solvent. Relevant moments of the total claim amounts are assumed to be finite and the total claim amounts of the companies under merging are assumed to be independent.

3. The occurrence process of the claims is a Poisson process with the intensity function λ. Assume that λ is positive, continuous and increasing, and that $\lim_{t\to\infty} \lambda(t) = m \in (0, \infty)$. The claim sizes all equal 1. The reporting delays are i.i.d. random variables with the distribution function G and they are independent of the occurrence process. The expectation of the reporting delay is $\varrho \in (0, \infty)$. The compensations are paid at the reporting times. The company starts the insurance business at time 0.

Let U_n be the outstanding claims at time n. Prove that $\lim_{n\to\infty} \mathbb{E}(U_n) = m\varrho$.

4. The yearly net payouts $\xi_1, \xi_2, \ldots$ of the company are i.i.d. random variables. Denote by c be the cumulant generating function of ξ_1. Assume that c is finite everywhere and that $\lim_{s\to\infty} c'(s) = \infty$. Assume further that the equation $c(s) = 0$ has a unique positive root R. Let $0 < y < x < 1/c'(R)$.

Let U_0 be the initial capital of the company and let T be the time of ruin. Prove that

$$\lim_{U_0\to\infty} U_0^{-1} \log \mathbb{P}\left(T \in (yU_0, xU_0]\right) = -xc^*(1/x).$$

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1. $\mathbb{P}(K_1 = k_1) = \sum_{k_2=0}^{\infty} \mathbb{P}(K_1 = k_1, K_2 = k_2)$

$$= \int_0^\infty e^{-\lambda_1 q} \frac{(\lambda_1 q)^{k_1}}{k_1!} \underbrace{\sum_{k_2=0}^{\infty} e^{-\lambda_2 q} \frac{(\lambda_2 q)^{k_2}}{k_2!}}_{=1} dH(q)$$

$$= \int_0^\infty e^{-\lambda_1 q} \frac{(\lambda_1 q)^{k_1}}{k_1!} dH(q), \text{ mixed Poisson} - (\lambda_1, Q),$$

$$\mathbb{P}(K_1 + K_2 = k) = \sum_{k_1+k_2=k} \int_0^\infty e^{-\lambda_1 q} \frac{(\lambda_1 q)^{k_1}}{k_1!} e^{-\lambda_2 q} \frac{(\lambda_2 q)^{k_2}}{k_2!} dH(q)$$

$$= \int_0^\infty e^{-(\lambda_1+\lambda_2) q} \frac{((\lambda_1+\lambda_2)q)^{k_1+k_2}}{(k_1+k_2)!} dH(q)$$

by the additivity property of Poisson distributions,

Thus K_1+K_2 is mixed Poisson $-(\lambda_1+\lambda_2, Q)$

2. Lecture notes, Section 6.2.3.

3. Theorem 8.3.1,

$$\mathbb{E}(U_n) = \int_0^n \lambda(s)(1 - G(n-s))ds)$$

$$= \int_0^n \lambda(n-t)(1-G(t))\,dt.$$

Clearly,

$$\mathbb{E}(U_n) \le m \int_0^n (1-G(t))\,dt \to m\rho, \quad n \to \infty.$$

For a given $\varepsilon > 0$, take $M > 0$ such that $\lambda(t) \ge m - \varepsilon$ for every $t \ge M$. Then

$$\mathbb{E}(U_n) \ge \int_0^{n-M} (m-\varepsilon)(1-G(t))\,dt \to (m-\varepsilon)\rho, \quad n \to \infty$$

4. $\mathbb{P}(T \in (yU_0, xU_0])$

$= \mathbb{P}(T \leq xU_0) - \mathbb{P}(T \leq yU_0)$

$= \mathbb{P}(T \leq xU_0)\left(1 - \dfrac{\mathbb{P}(T \leq yU_0)}{\mathbb{P}(T \leq xU_0)}\right).$

By Theorem 9.12,

$$\lim U_0^{-1} \log \mathbb{P}(T \leq xU_0) = -xc^{\alpha}\left(\tfrac{1}{x}\right),$$

$$\lim U_0^{-1} \log \mathbb{P}(T \leq yU_0) = -yc^{\alpha}\left(\tfrac{1}{y}\right).$$

By Lemma 9.1.1, $yc^{\alpha}\left(\frac{1}{y}\right) > xc^{\alpha}\left(\frac{1}{x}\right)$ so that for a given $\varepsilon > 0$,

$$\mathbb{P}(T \leq yU_0) \leq e^{-\left(yc^{\alpha}\left(\frac{1}{y}\right) - \varepsilon\right)U_0},$$

$$\mathbb{P}(T \leq xU_0) \geq e^{-\left(xc^{\alpha}\left(\frac{1}{x}\right) + \varepsilon\right)U_0}$$

for large U_0. Choose ε such that

$$xc^{\alpha}\left(\tfrac{1}{x}\right) + \varepsilon < yc^{\alpha}\left(\tfrac{1}{y}\right) - \varepsilon$$

to see that

$$\lim_{U_0 \to \infty} \frac{\mathbb{P}(T \leq yU_0)}{\mathbb{P}(T \leq xU_0)} = 0.$$

This proves the claim.