Department of Mathematics and Statistics
Riemannian Geometry
Final exam
16.5.2013

1. Let $\gamma\colon I \to \mathbb{R}^n$ be a C^∞-path. Show that a vector field $V \in \mathcal{T}(\gamma)$, $V = (V^1, \dots, V^n)$, is parallel with respect to the Euclidean connection if and only if its components V^i are constant functions.

2. (a) Prove that the Lie bracket $\mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$,

$$(X, Y) \mapsto [X, Y]$$

is not a connection.

(b) Prove that there exist smooth vector fields $V \in \mathcal{T}(\mathbb{R}^2)$ and $W \in \mathcal{T}(\mathbb{R}^2)$ such that $V = W = \frac{\partial}{\partial x}$ along the x-axis, but the Lie brackets $\left[V, \frac{\partial}{\partial y}\right]$ and $\left[W, \frac{\partial}{\partial y}\right]$ are not equal on the x-axis.

3. Let M and N be Riemannian manifolds and $f\colon M \to N$ a diffeomorphism. Suppose that N is complete and that there exists a constant $c > 0$ such that

$$|v| \geq c|f_{*p}v|$$

for all $p \in M$ and for all $v \in T_pM$. Prove that M is complete.

4. Let $\gamma\colon [a, b] \to M$ be a geodesic and V a Jacobi field along γ. Prove that

$$\langle V_t, \dot{\gamma}_t \rangle = \langle V_a, \dot{\gamma}_a \rangle + (t - a)\langle V'_a, \dot{\gamma}_a \rangle$$

for every $t \in [a, b]$.

5. (a) Formulate the Rauch comparison theorem.

(b) Let M be a complete Riemannian manifold such that $K(\sigma) \leq 0$ for every 2-dimensional subspace $\sigma \subset T_pM$ and for every $p \in M$. Prove that $\exp_p\colon T_pM \to M$ is a local diffeomorphism for every $p \in M$.