(MAST30173) Partial differential equations 2

Exam

8 May 2024

Problem 1

Consider the initial-boundary value problem for the *vibrations of a membrane with fixed edges*

$$\begin{cases} u_{tt} - c^2\Delta u = 0, & Q \times (0,\infty) \\ u = f,\ u_t = g, & Q \times \{t = 0\} \\ u = 0, & \partial Q \times (0,\infty) \end{cases},$$

where $Q := [0,a] \times [0,b] \subset \mathbb{R}^2$ is a fixed rectangle of sides $a, b > 0$, and $f, g \in C^\infty(Q)$. Use separation of variables to find simple solutions satisfying the boundary conditions, then use the principle of superposition to find solutions which satisfy the initial conditions as well.

Problem 2

Consider the initial value problem for the one-dimensional *telegraph equation*

$$\begin{cases} u_{tt} + 2\sigma u_t - u_{xx} = 0, & \mathbb{R} \times (0,\infty) \\ u = g,\ u_t = h, & \mathbb{R} \times \{t = 0\} \end{cases},$$

where $g, h \in L^2(\mathbb{R})$ and $\sigma > 0$ is a fixed constant. Use the Fourier transform method to find a representation formula for the solution $u(x,t)$. How does the solution behave as $t \to \infty$?

Problem 3

Show that the function

$$u(x) := \log|1 - \log|x||$$

belongs to $H^1(B_1) \setminus C^0(B_1)$, where $B_1 \subset \mathbb{R}^2$ is the open unit ball.

[*Hint: the inequality* $\log \rho \leq \rho - 1$ *holds for all* $\rho > 0$.]

Problem 4

Consider the problem

$$\begin{cases} \Delta u = 0, & \Omega \\ u = 0, & \partial\Omega \end{cases},$$

where $\Omega \subset \mathbb{R}^2$ is the open set

$$\Omega := \left\{ (r, \theta) : r \in (0, 1), \theta \in \left(0, \frac{3\pi}{2}\right) \right\}.$$

Show that the function

$$v(r, \theta) := \left(\frac{1}{r^{2/3}} - r^{2/3} \right) \sin\left(\frac{2\theta}{3}\right)$$

belongs to $W_0^{1,p}(\Omega)$ for all $p \in \left(1, \frac{6}{5}\right)$, and that v is a nontrivial weak solution to the above problem. Moreover, give a reason why uniqueness of solutions may not hold in this case.

[*Hint: the following relations between Cartesian and polar derivatives*

$$u_r = u_x \cos\theta + u_y \sin\theta,$$

$$u_\theta = -ru_x \sin\theta + ru_y \cos\theta,$$

$$\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$$

are known to hold.]

Problem 5

Use a counterexample to show that the Calderón - Zygmund estimate for the regularity of solutions to the Poisson equation $\Delta u = f$ in a bounded set does not hold in the limit case $p = \infty$.