Model theory
Course exam 10.12.2024

Answer 4 of the following 5 questions.

Question 1.

(a) Give the definition of elementarily equivalent L-structures.

(b) Give examples of L-structures $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{M} \equiv \mathcal{N}$ but $\mathcal{M} \not\cong \mathcal{N}$ (and justify your answer).

(c) Prove the Tarski-Vaught test: let $\mathcal{M}$ and $\mathcal{N}$ be L-structures with $M \subseteq N$. Then $\mathcal{M} \preceq \mathcal{N}$ if and only if for all L-formulas $\phi(x, \overline{y})$ and $\bar{a} \in M^{|\overline{y}|}$, if there is $b \in N$ so that $\mathcal{N} \models \phi(b, \overline{a})$ then there is $c \in M$ so that $\mathcal{N} \models \phi(c, \overline{a})$.

Question 2.

(a) Let T be an L-theory. Suppose that for any quantifier free formula $\phi(\bar{x}, a)$ there is a quantifier free formula $\psi(\bar{x})$ such that $T \models \exists a \phi(\bar{x}, a) \leftrightarrow \psi(\bar{x})$. Prove that T has quantifier elimination.

(b) Consider $L = \{E\}$, where E is a binary relation symbol. Let T_0 be the L-theory of equivalence relations and $T = T_0 \cup \{\phi_n, \psi_n : n \geqslant 2\}$, where

$$\phi_n := \exists x_1, \ldots, \exists x_n \bigwedge_{i<j\leqslant n} \neg E(x_i, x_j),$$

and

$$\psi_n := \forall x \exists x_1, \ldots, \exists x_n \bigwedge_{i<j\leqslant n} x_i \neq x_j \wedge \bigwedge_{i}^{n} E(x, x_i).$$

Show that T has quantifier elimination.

In questions 3-5, let T be a complete theory in a countable language with infinite models, and let κ be an infinite cardinal.

Question 3.

(a) Give examples of L-structures $\mathcal{M}, \mathcal{N} \models T$ such that $\mathcal{M}$ is $|M|$-saturated and $\mathcal{N}$ is not $|N|$-saturated (and justify your answer).

(b) Show that if $\mathcal{M} \models T$ is κ-saturated then it is κ^+-universal.

Question 4.

(a) Let $A \subset \mathcal{M} \models T$. What does it mean that a type $p \in S_n^{\mathcal{M}}(A)$ is isolated?

(b) Recall that $\mathcal{M} \models T$ is atomic if $tp(\overline{a})$ is isolated for all tuples $\overline{a}$ of elements of M. Give an example of an atomic L-structure.

(c) Show that if $\mathcal{M} \models T$ is countable and atomic then $\mathcal{M}$ is a prime model of T.

Question 5.

(a) What does it mean that T is κ-stable?

(b) Give examples of L-theories T' and T'' such that T' is ω-stable and T'' is not ω-stable.

(c) Show that if T is ω-stable then it is κ-stable for each infinite cardinal κ.