Matematiikan ja tilastotieteen osasto
Mitta ja integraali
20.10.2020

1. Olkoon $A \subset \mathbb{R}$ 0-mittainen eli $m(A) = 0$. Osoita, että
$$m\big(\{e^x \colon x \in A\}\big) = 0.$$

2. Osoita, että joukko
$$A = \{(x, y) \in \mathbb{R}^2 \colon x \geq 1,\ 0 < y < 1/x^2\}$$
on mitallinen ja että $m_2(A) < \infty$.

3. (a) Olkoon $A \subset \mathbb{R}^n$ ja $f \colon A \to \mathbb{R}$ ja $g \colon A \to \mathbb{R}$ mitallisia funktioita. Osoita, että joukot
$$\{x \in A \colon f(x) < g(x)\} \quad \text{ja} \quad \{x \in A \colon f(x) = g(x)\}$$
ovat mitallisia.
   (b) Olkoot $f_k \colon A \to \mathbb{R},\ k \in \mathbb{N}$, mitallisia funktioita. Osoita, että joukko
$$\{x \in A \colon \text{on olemassa raja-arvo } \lim_{k\to\infty} f_k(x)\}$$
on mitallinen.

4. Laske raja-arvo
$$\lim_{k\to\infty} \int_1^\infty e^{-k/\sqrt{x}} \cos(k + x^2)\, dx.$$

5. Olkoon $f \colon \mathbb{R} \to \mathbb{R}$ integroituva. Osoita, että
$$\liminf_{k\to\infty} \int_k^{k+1} |f| = 0.$$

Department of Mathematics and Statistics
Measure and integral
20.10.2020

1. Let $A \subset \mathbb{R}$ be of measure 0, $m(A) = 0$. Prove that
$$m\big(\{e^x : x \in A\}\big) = 0.$$

2. Prove that the set
$$A = \{(x,y) \in \mathbb{R}^2 : x \geq 1,\ 0 < y < 1/x^2\}$$
is measurable and that $m_2(A) < \infty$.

3. (a) Let $A \subset \mathbb{R}^n$ and $f\colon A \to \mathbb{R}$ and $g\colon A \to \mathbb{R}$ be measurable functions. Prove that the sets
$$\{x \in A : f(x) < g(x)\} \quad \text{and} \quad \{x \in A : f(x) = g(x)\}$$
are measurable.
   (b) Let $f_k\colon A \to \mathbb{R},\ k \in \mathbb{N}$, be measurable functions. Prove that the set
$$\{x \in A : \text{the limit } \lim_{k\to\infty} f_k(x) \text{ exists}\}$$
is measurable.

4. Find the limit
$$\lim_{k\to\infty} \int_1^\infty e^{-k/\sqrt{x}} \cos(k + x^2)\, dx.$$

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be integrable. Prove that
$$\liminf_{k\to\infty} \int_k^{k+1} |f| = 0.$$