Department of Mathematics and Statistics
Measure and integration
Exam 28 February 2024

1. Find the limit and justify your answer:
$$\lim_{k\to+\infty}\int_0^{+\infty} e^{-x}\left(1+(\sin(1/x))^k\right)^3\,dx$$

2. Assume that X is an arbitrary set and $\Gamma \subset \mathcal{P}(X)$ is a σ-algebra. Let μ be a measure on Γ, assume that $\mu(X) < \infty$. Let $A_1, A_2, \ldots \in \Gamma$ be some sets and assume that for any $k \geq 1$ we have
$$\mu(\bigcap_{i=1}^{k} A_i) \geq 1.$$
Prove that $\bigcap_{i=1}^{+\infty} A_i \neq \varnothing$.

3. Recall that $\mathfrak{m}_n^*$ denotes the outer measure. Assume that $A \subset \mathbb{R}^n$ is such that $\mathfrak{m}_n^*(A) < \infty$ and there exists a (Lebesgue) measurable set $A_1 \subset A$ such that $\mathfrak{m}_n(A_1) = \mathfrak{m}_n^*(A)$. Find $\mathfrak{m}_n^*(A \setminus A_1)$. Prove that A is measurable.

4. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is an integrable function. Given $\varepsilon > 0$ define
$$E_\varepsilon = \{x \in \mathbb{R}^n \mid |f(x)| \leq \varepsilon\}.$$
Prove that for any $\varepsilon > 0$ the set E_ε is measurable $\mathfrak{m}_n(E_\varepsilon) = +\infty$. Prove that
$$\lim_{\varepsilon\to 0}\int_{E_\varepsilon} f = 0.$$