Mathematics of infectious diseases
EXAM: 22 October 2025

1. *HPV and cancer.* An HPV infection goes through several stages before causing cancer. Suppose there are three states, HPV-positive (state 1), pre-cancerous (state 2) and cancerous (state 3; in reality, there are multiple pre-cancerous states). An infection in state $i(=1,2)$ progresses to state $i+1$ in a fixed time T_i; during this time, it clears to full recovery at a constant rate ν_i. From cancer (state 3), there is no spontaneous recovery. Individuals with cancer are diagnosed at a rate δ and die due to the disease at a rate α. All individuals have a background death rate μ. Calculate the probability that a new infection in state 1 ends with diagnosed cancer.

2. *Final size.* Suppose that infecteds go through two phases of the disease: First, they are highly infectious for a fixed time interval T, infecting a susceptible upon contact with probability p_1. Following this, there is a second time period for which the probability of infecting a susceptible upon contact is p_2 (lower than p_1). From this second phase, recovery occurs at a constant rate ν. The population is well mixed and the contact rate is constant at c irrespectively of infection status and population size.

(a) Does it matter for the final size whether some of the infected die (at a rate ν) rather than recover at the end of the second phase?
(b) Give the equation that determines the final size in terms of the parameters given above. (If it matters, assume no death.)

3. *The SEIR model.* Consider the SEIR model including natural deaths at a rate μ and deaths due to the disease at a rate α,

$$\begin{aligned}
\frac{dS}{dt} &= B - \beta SI - \mu S \\
\frac{dE}{dt} &= \beta SI - \theta E - \mu E \\
\frac{dI}{dt} &= \theta E - \nu I - \mu I - \alpha I \\
\frac{dR}{dt} &= \nu I - \mu R
\end{aligned}$$

Calculate R_0 and the probability of a major outbreak if started with a single host in the

exposed (infected but not yet infectious) state E assuming that the disease-free population is at its equilibrium.

4. *Vaccination of wild animals.* In a population of wild animals, it is not possible to keep track of which individual has been vaccinated and which has not been. The best we can do is that we vaccinate randomly chosen individuals at a *per capita* rate v. The vaccine has no effect if the individual is already immune. Vaccinated individuals lose their immunity at a rate θ. Denoting the number of susceptibles and of the vaccinated immune individuals respectively by S and P (from 'protected'), the disease-free population follows the dynamics

$$\begin{aligned} \frac{dS}{dt} &= B - \mu S - vS + \theta P \\ \frac{dP}{dt} &= vS - \mu P - \theta P \end{aligned}$$

where B is the population birth rate and μ is the *per capita* death rate.

(a) Find the equilibrium $(\bar{S}, \bar{P})$ and show that the equilibrium is stable.
(b) What is the minimum vaccination rate v necessary to prevent the outbreak of a disease with a given R_0?