Department of Mathematics and Statistics
Introduction to differential geometry
2nd Exam

1. Let $X, Y \in \mathcal{T}(\mathbb{R}^3)$ be smooth vector fields defined by

$$X_p = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} \quad \text{and} \quad Y_p = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}$$

for $p = (x, y, z) \in \mathbb{R}^3$.

(a) Find the maximal integral curve of X starting at a point $p_0 = (x_0, y_0, z_0)$.

(b) Compute the Lie derivative $L_X Y$ directly by using the definition (3.61) in the lecture notes. Verify that $L_X Y = [X, Y]$ as it should be.

2. Let $X \in \mathcal{T}(\mathbb{R}^2 \setminus \{0\})$ be the smooth vector field

$$X = \frac{x}{x^2 + y^2}\frac{\partial}{\partial x} + \frac{y}{x^2 + y^2}\frac{\partial}{\partial y}$$

and let $\omega \in \mathcal{A}^2(\mathbb{R}^2)$ be the differential 2-form $\omega = dx \wedge dy$. Compute $L_X \omega$ in $\mathbb{R}^2 \setminus \{0\}$. How do you interpret this (in your own words)?

3. Prove Corollary 7.15 (Integration by parts): Let M be a smooth, compact, oriented n-manifold with boundary, $X \in \mathcal{T}(M)$, $\alpha \in \mathcal{A}^k(M)$, and $\beta \in \mathcal{A}^{n-k}(M)$, $0 \leq k \leq n$. Then

$$\int_M (L_X \alpha) \wedge \beta = \int_{\partial M} i_X(\alpha \wedge \beta) - \int_M \alpha \wedge (L_X \beta).$$