Department of Mathematics and Statistics
Introduction to differential geometry
Exam

1. Suppose that M and N are differentiable manifolds and that M is connected.

(a) Let $f\colon M \to N$ be a smooth mapping such that

$$f_{*p}\colon T_pM \to T_{f(p)}N$$

is the zero mapping for every $p \in M$, that is, $f_{*p}v = 0 \in T_{f(p)}N$ for every $p \in M$ and for every $v \in T_pM$. Show that f is a constant mapping.

(b) What is strange in the following claim? If $f\colon M \to N$ and $g\colon M \to N$ are smooth mappings such that $f(m) = g(m)$ for some $m \in M$ and $f_{*p} = g_{*p}$ for all $p \in M$, then $f = g$.

2. Let $F\colon \mathbb{R}^2 \to \mathbb{R},\ F(x,y) = x^2+4y^4-4y^2$, and denote $M_t = \{(x,y) \in \mathbb{R}^2\colon F(x,y) = t\}$ for $t \in \mathbb{R}$. Find all values of $t \in \mathbb{R}$ such that M_t is a submanifold of $\mathbb{R}^2$.

3. Let $M = \mathrm{GL}(2,\mathbb{R}),\ \mathcal{D} = \mathbb{R} \times M$ and $\theta\colon \mathcal{D} \to M$,

$$\theta(t,A) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} A,$$

where the right-hand side is the matrix product. Prove that θ is a flow and find its infinitesimal generator.