University of Helsinki Kankaanrinta
Department of Mathematics and Statistics

Introduction to Algebraic Topology, MAST 31023

Final Exam, Dec 11 (general exam, no calculators)

(1) (6 points) Let $A = \{(x, y) \in \mathbb{S}^n \times \mathbb{S}^n \mid x \neq y\}$, where $n \geq 0$. Let

$$f \colon \mathbb{S}^n \to A, \quad x \mapsto (x, -x).$$

Show that f is a homotopy equivalence.

(2) (6 points)

(a) Let $h \colon \mathbb{S}^1 \to \mathbb{S}^1$ be nullhomotopic. Show that h has a fixed point.

(b) Let A be a subspace of $\mathbb{R}^n$, and let Y be a topological space. Let $a_0 \in A$ and $y_0 \in Y$. Let $h \colon A \to Y$ be a continuous map with $h(a_0) = y_0$. Show that if h is extendable to a continuous map $\mathbb{R}^n \to Y$, then $h_* \colon \pi_1(A, a_0) \to \pi_1(Y, y_0)$ is the trivial homomorphism.

(3) (6 points) Let X be a topological space and $x_0 \in \mathbb{S}^n$. Show that

$$H_i(X \times \mathbb{S}^n) \cong H_i(X) \oplus H_i(X \times \mathbb{S}^n, X \times \{x_0\}),$$

for all $i \geq 0$.

(4) (6 points) Consider the following commutative diagram of abelian groups and group homomorphisms:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{i_1} & A & \xrightarrow{\pi_1} & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\
0 & \longrightarrow & B' & \xrightarrow{i_2} & B & \xrightarrow{\pi_2} & B'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \\
0 & \longrightarrow & C' & \xrightarrow{i_3} & C & \xrightarrow{\pi_3} & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Assume all the horizontal rows are exact, and assume that the left and the middle columns are exact. Show that column on the right is exact at B''. (In fact the column

$$0 \to A'' \overset{\varphi_3}{\to} B'' \overset{\psi_3}{\to} C'' \to 0$$

on the right is an exact sequence but you do **not** have to prove the exactness at A'' and C''.)

(5) (6 points) Let X be the subspace of $\mathbb{R}^3$ that is the union of the standard unit sphere $\mathbb{S}^2$, the circle $\{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = \frac{1}{4}\}$ and the two line segments $\{(x, 0, 0) \mid \frac{1}{2} \leq x \leq 1\}$ and $\{(x, 0, 0) \mid -1 \leq x \leq -\frac{1}{2}\}$. Use a Mayer-Vietoris sequence to calculate the homology groups of X.