Department of Mathematics and Statistics / UH
MAST30171 Functional Analysis II - Period II - 2025

Final exam on Dec. 17th 2025, at 9.00-12.00, auditorium D123

You are not allowed to use a calculator, tables or other written material except for one page of A4 size containing notes of your own choice, max. about 60 lines × 100 characters / line.

In all of the problems, the scalar field $\mathbb{K}$ of the Banach or Hilbert spaces is assumed to be the field of real numbers $\mathbb{R}$.

1. Let ℓ^2 be the Hilbert space of square summable sequences of real numbers. Give an example of two continuous non-commuting, bounded linear operators $S : \ell^2 \to \ell^2$ and $T : \ell^2 \to \ell^2$, in other words, operators with $ST \neq TS$.

2. Let $1 < p < \infty$ and $1/p + 1/q = 1$.
a) The map $G : x = (x_k)_{k=1}^\infty \mapsto 2x_2 - 4x_3 + 6x_5$ is continuous and linear $\ell^p \to \mathbb{K}$, in other words, an element of the dual of ℓ^p. Which element $y = (y_k)_{k=1}^\infty$ of ℓ^q corresponds to G in the identification $(\ell^p)^* = \ell^q$ (meaning, there holds $\langle x, y \rangle = Gx$ for all x)?
b) Let $y = (y_n)_{n=1}^\infty$ be a sequence of numbers with $y_n = 2^{-n}$ for all $n \in \mathbb{N}$. Show, how the sequence y defines an element of the dual of ℓ^p, where $p = 4$.
c) For y as in problem b), calculate the number $\langle x, y \rangle$, when $x = (x_n)_{n=1}^\infty \in \ell^4$ is a sequence with $x_n = 1$ for $1 \leq n \leq 10$, and $x_n = 0$ for $n > 10$.

3. Are the following bilinear mappings S continuous or coercive $\ell^2 \times \ell^2 \to \mathbb{R}$:

$$\text{a) } S : (x, y) \mapsto \sum_{k=1}^\infty x_k y_{k+1}, \quad \text{b) } S : (x, y) \mapsto \sum_{k=1}^\infty 2^{-k} x_k y_k,$$

$$\text{c) } S : (x, y) \mapsto \sum_{k=1}^\infty x_k y_k - \frac{2}{3} \sum_{k=1}^\infty x_k y_{k+1},$$

where $x = (x_k)_{k=1}^\infty \in \ell^2$, $y = (y_k)_{k=1}^\infty \in \ell^2$? (*Recall:* continuity = boundedness. Coercivity: $S(x, x) \geq C\|x\|^2$.)

4. Here *you need to answer to only one of the problems a) or b)*; the choice is yours!
a) Write a short proof for the fact that the space $C(0, 1)$ endowed with the L^1-norm

$$\|f\|_1 = \int_0^1 |f(t)| dt$$

cannot be complete. (You should use the results presented on the lectures, for example the Open Mapping Theorem.)

b) Show that the Banach space $C^1(0, 1)$ of continuously differentiable functions on $[0, 1]$ is a Banach algebra with respect to the pointwise multiplication, when endowed with the norm

$$\|f\| := \sup_{x \in [0,1]} \{|f(x)|\} + \sup_{x \in [0,1]} \{|f'(x)|\}.$$