Department of Mathematics and Statistics / UH
MAST30170 Functional Analysis I - Period I - 2024

Exam on Oct. 23rd, 2024 at 9.00-12.00, auditorium CK112

You are not allowed to use a calculator, tables or other written material except for one page of A4 size containing notes of your own choice, max. about 60 lines × 100 characters / line

1. Do the following sequences $(f_n)_{n=1}^{\infty}$ converge in the Banach-space $C(0,1)$ (continuous functions on the interval $[0,1]$ with sup-norm; $t \in [0,1]$ is a variable):

$$\text{a)}\ \ f_n(t) = \frac{1}{t+10+n}\ ,\quad \text{b)}\ \ f_n(t) = \frac{1}{t+10+\frac{1}{n}}.$$

2. Let E be a Hilbert space. By using the elementary properties of the inner product and norm, prove the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big),$$

where $x, y \in E$ are arbitrary.
Using the parallelogram identity, prove that the Banach space $L^p(-1,1)$ (which consists of L^p–integrable functions on the interval $(-1,1)$) is not a Hilbert space, when $p = 4$.

3. Show that the integral equation

$$5e^{-t} + \int_0^1 \frac{e^{-s}}{10+t} f(s)ds - f(t) = 0\ \ ,\ t \in [0,1],$$

has a unique solution in the space $C(0,1)$. You may assume that here, the integral expression is a continuous function of t for all $f \in C(0,1)$.

4. Let $(x_n)_{n=1}^{\infty}$ be a sequence in the Hilbert–space H. We say that it converges *weakly* in H to the vector $x \in H$, if for all $y \in H$ we have

$$(x_n|y) \to (x|y)\ ,\ \text{as } n \to \infty,$$

where the convergence happens in $\mathbb{K}$.
(i) Show that convergence in the conventional sense implies weak convergence.
(ii) Let $H := \ell^2$ be the Hilbert space of square summable sequences and let $e_n \in \ell^2$ be the usual nth basis vector, i.e.

$$e_n = (0, \dots, 0, 1, 0, 0, \dots)$$

where the nth coordinate is equal to 1 and the others are 0. Show that $(e_n)_{n=1}^{\infty}$ converges to $0 = (0,0,0,\dots) \in \ell^2$ weakly.