Fourier Analysis II – Spring 2019

Course exam 6.5.2019 at 9:15–11:45 ($2\frac{1}{2}$ hours)

Solve any **four (4)** of the following problems. (The choice is yours, all are worth the same.)

In any problem that you choose, it is always a good idea to write down the definitions of the objects and symbols appearing in the statement of the problem: this can earn you some points, even if you don't solve the full problem. It is also a good idea to say, if you use some non-trivial theorem. In general you are allowed to use any results from the course, except of course the ones that you are specifically asked to prove.

Problem 1. Let $u \in L^1(\mathbb{R})$, and suppose that also the function $x \mapsto |x| \cdot u(x)$ belongs to $L^1(\mathbb{R})$. Prove that the Fourier transform $\hat{u}$ is differentiable at every point.

Problem 2. Prove the formulas

$$\widehat{f * g} = \hat{f}\hat{g}, \qquad \widehat{fg} = \hat{f} * \hat{g}$$

for all $f, g \in \mathscr{S}(\mathbb{R}^d)$. You can use other known properties about the Fourier transform on $\mathscr{S}(\mathbb{R}^d)$, as long as you state what you use.

Problem 3. Prove the formula

$$\int \hat{f} g = \int f \hat{g}$$

for all $f, g \in \mathscr{S}(\mathbb{R}^d)$. Then choose a suitable g to verify Plancherel's formula $\|\hat{f}\|_2 = \|f\|_2$ for all $f \in \mathscr{S}(\mathbb{R}^d)$. You can use other known properties about the Fourier transform on $\mathscr{S}(\mathbb{R}^d)$, as long as you state what you use.

Problem 4. The function $e^{-\pi|x|^2}$ has a special role in Fourier analysis. Explain briefly why. (Just state the relevant fact, no need to prove it.) Let

$$h^t(x) := \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \qquad x \in \mathbb{R}^d, \quad t > 0,$$

and define the *heat semigroup* $T_t f := h^t * f$ for $f \in L^1(\mathbb{R}^d)$. Prove that this satisfies the *semigroup property*

$$T_t(T_s f) = T_{t+s} f, \qquad \forall f \in L^1(\mathbb{R}^d), \quad \forall t, s > 0.$$

Problem 5. *Explain* in detail the meaning of the formula

$$\Big(\sum_{n \in \mathbb{Z}} \delta_n\Big)^\wedge = \sum_{n \in \mathbb{Z}} \delta_n.$$

(What is δ_n, in what sense does the series converge, what is meant by the Fourier transform of such a series, etc.) You *don't need to prove* the formula.