Department of Mathematics and Statistics
Elements of Set Theory, Spring 2025

Exam
5/5/2025

Answer all 4 questions.

1. (a) Let κ, λ be cardinal numbers. Define $\kappa \cdot \lambda$ and κ^λ (in cardinal arithmetic).

   (b) Show directly from the definitions of cardinal arithmetic that for all cardinals κ, λ, μ,
   $$(\kappa^\lambda)^\mu = \kappa^{\mu \cdot \lambda}.$$

2. Define for every $f, g \in \mathbb{N}^\mathbb{N}$:
   $$f <^* g \iff \exists m \in \mathbb{N}\, \forall n > m\ (f(n) < g(n))$$

   (a) Prove that $<^*$ is a (strict) partial ordering.

   (b) Is $<^*$ a linear ordering?

   (c) Find a set $A \subseteq \mathbb{N}^\mathbb{N}$ such that $\langle A, <^* \rangle$ is isomorphic to $\langle \omega, \in \rangle$. Prove your claim.

3. (a) Define what is an ordinal.

   (b) Let A be a set of ordinals. Prove that $\alpha = \bigcup A$ is also an ordinal.
   You may use results proven in class, as long as you state them (and as long as they don't rely on what needs to be proven).

4. Let $\langle A, \preceq \rangle$ be a partially ordered set.

   We say that a set $C \subseteq A$ is:

   - a *chain* if $\forall x, y \in C\ (x \preceq y \vee y \preceq x)$;
   - *cofinal* if $\neg \exists a \in A \forall c \in C\ (c \prec a)$.

   Prove, assuming the Axiom of Choice, that there is an cofinal chain $C \subseteq A$.

   State precisely the form of the Axiom of Choice you are using.