Bayesian inference, Course Exam 9.5.2019

Exam time: 2h 30min
Permitted accessories: -
For this exam, extra points for exercises are available from the Spring 2019 course.

1. Assume i.i.d sample $Y_1, \ldots, Y_n$ from the Pareto distribution $\text{Pareto}(\alpha, x_0)$, where the scale parameter x_0 is a known constant. A conjugate prior for the shape parameter α is a gamma distribution; now the full model is

$$Y_i \,|\, \alpha \sim \text{Pareto}(\alpha, x_0) \quad \text{for all } i = 1, \ldots, n$$
$$\alpha \sim \text{Gamma}(a, b).$$

Derive the posterior distribution $p(\alpha \,|\, \mathbf{y})$ for the parameter α given all the observations $\mathbf{y} = (y_1, \ldots, y_n)$.

2. Consider one new observation $\tilde{Y} \,|\, \alpha \sim \text{Pareto}(\alpha, x_0)$ from the same distribution as the observations of the previous exercise. Assume also that this new observation is conditionally independent from the original observations given the parameter α. Denote the parameters of the posterior distribution of the previous exercise as a_n and b_n (assume the same gamma prior $\text{Gamma}(a, b)$ from the previous exercise).

Show that the posterior predictive distribution for the new observation can be written as:

$$p(\tilde{y} \,|\, \mathbf{y}) = \frac{b_n^{a_n} a_n}{\tilde{y} \left(b_n + \log \frac{\tilde{y}}{x_0} \right)^{a_n + 1}}$$

3. Assume i.i.d. observations $Y_1, \ldots, Y_n$ from the gamma distribution with both parameters α and β *unknown*:

$$Y_i \,|\, \alpha, \beta \sim \text{Gamma}(\alpha, \beta) \quad \text{for all } i = 1, \ldots, n.$$

The conjugate prior for this model with hyperparameters $p, r, q, s > 0$ is of the form

$$p(\alpha, \beta \,|\, p, q, r, s) \propto \frac{\beta^{\alpha s}}{\Gamma(\alpha)^r} p^{\alpha - 1} \exp(-\beta q), \quad \text{where } \alpha, \beta > 0.$$

(a) Show that given this prior, the joint posterior for the parameters α and β given all the observations has the same form:

$$p(\alpha, \beta \,|\, \mathbf{y}) \propto \frac{\beta^{\alpha s_n}}{\Gamma(\alpha)^{r_n}} p_n^{\alpha - 1} \exp(-\beta q_n),$$

where the parameters of the posterior distribution are

$$p_n := p \prod_{i=1}^{n} y_i,$$
$$q_n := q + \sum_{i=1}^{n} y_i,$$
$$r_n := r + n,$$
$$s_n := s + n.$$

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(b) Show that the marginal posterior for the parameter α given all the observations $\mathbf{y} = (y_1, \ldots, y_n)$ is of the form:

$$p(\alpha \mid \mathbf{y}) \propto \frac{\Gamma(\alpha s_n + 1)}{\Gamma(\alpha)^{r_n} q_n^{\alpha s_n + 1}} \, p_n^{\alpha - 1}, \quad \text{where } \alpha > 0.$$

4. Explain briefly meanings of the following terms (exact mathematical definitions not necessarily needed), especially from the viewpoint of Bayesian inference:

(a) Bayes factor

(b) A highest posterior density (HPD) region

(c) Monte Carlo Markov chain (MCMC) methods

(d) Improper prior

(e) Empirical Bayes

Density functions

- The density function of the random variable X following the **Pareto distribution** $\mathrm{Pareto}(\alpha, x_0)$ is
$$p(x \mid \alpha, x_0) = \frac{\alpha x_0^\alpha}{x^{\alpha+1}}, \quad \text{when } x \geq x_0,$$
where $\alpha, x_0 > 0$.

- The density function of the random variable X following the **gamma distribution** $\mathrm{Gamma}(\alpha, \beta)$ is:
$$p(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x), \quad \text{when } x > 0,$$
where $\alpha, \beta > 0$.

Integrals

- Gaussian integral:
$$\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x = \sqrt{\pi}.$$

- Gamma integral:
$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t.$$

- Beta function:
$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1} \, \mathrm{d}t.$$

- Properties of the gamma integral:
  -
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$
  - for all $x > 0$ it holds that:
$$\Gamma(x+1) = x\Gamma(x).$$