ALGEBRAIC STRUCTURES II EXAM, MAY 3 2024

1) (6+8=14 pts)

(a) Define the following concepts/structures: "An ideal of a ring R" and "A group action of G on the set X".
(b) Arrange (without proof) the following structures according to the hierarchy below: Integral domains, Principal ideal domains, Euclidean domains, Unique factorisation domains;

$$\text{Fields } \subset \ldots \subset \ldots \subset \ldots \subset \ldots$$

2) (15 pts)

Prove the following statement about polynomials. Given $f(x), g(x) \in \mathbb{R}[x]$, with $g(x) \not\equiv 0$, there exist unique polynomials $q(x), r(x) \in \mathbb{R}[x]$ such that

$$f(x) = q(x)g(x) + r(x), \qquad \text{with} \quad r(x) \equiv 0 \quad \text{or} \quad deg(r(x)) < deg(g(x)).$$

3) (15 pts)

Prove Cauchy's Theorem for Abelian groups: Let G be a finite Abelian group and p a prime number such that p divides $|G|$. Then G has an element of order p.

[Hint: Argue by induction on $|G|$ and consider the maximal proper subgroup H. You may use the following fact without proof. Let C_m be the cyclic group of order m. Then for any prime divisor $p|m$, C_m has a unique subgroup of order p.]

4) (6 pts) Is there a field with 10 elements ? Prove your claim.