MAT21010: ALGEBRAIC STRUCTURES I
UNIVERSITY OF HELSINKI
FINAL EXAM: 04TH-MARCH-2024
12:00–15:00

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Student name:

Final represents the 60% of the total course mark. The remainder 40% comes from the assignments.

Instructions:

(1) Students will not be allowed to begin an examination after it has been in progress for 30 minutes.
(2) Students must remain in the exam room until at least 30 minutes has elapsed.
(3) Write the solutions as clear as possible.
(4) In order to obtain full marks, it is very important to do the solutions with the material cover during the lectures and the content of the course.
(5) **Data, formula sheets, notes, calculator, and software components are strictly NOT allowed. Only the piece of paper of Assignment Seven is allowed!**

Problem 1: (16 marks)

Which of the following groups are cyclic? Justify your answer.

(i) For a fixed $n \in \mathbb{N}$, consider the roots of the unity $U_n := \{z \in \mathbb{C} : z^n = 1\}$ equipped with the usual multiplication of complex numbers. **4 marks.**
(ii) $(\mathsf{A}_n, \circ)$ being the alternating subgroup of the symmetric group $(\mathsf{S}_n, \circ)$ for $n \in \mathbb{N}\setminus\{1, 2, 3\}$. **4 marks.**
(iii) $(\mathsf{GL}(2, \mathbb{Q}), \cdot)$, where $\mathsf{GL}(2, \mathbb{Q})$ is the set of 2×2 matrices with rational entries and invertible equipped with the usual multiplication of matrices. **4 marks.**
(iv) $(\mathbb{Z}_p \setminus \{[0]\}, \cdot)$, where $\cdot$ is the usual multiplication mod p on $\mathbb{Z}_p$ and p is a positive prime number. **4 marks.**

Problem 2: (16 marks)

Let $n \in \mathbb{N}$ be fixed. Let $G = \mathsf{GL}(n, \mathbb{R})$ be the general linear group of degree n. More precisely, the group of all $n \times n$ invertible matrices with real coefficients embedded with the usual multiplication of matrices. We now consider the subset of G defined by

$$\mathsf{SL}(n, \mathbb{R}) = \{A \in G : \det(A) = 1\},$$

where det denotes the determinant function.

(i) Show that $\mathsf{SL}(n, \mathbb{R})$ is a normal subgroup of $\mathsf{GL}(n, \mathbb{R})$. **8 marks.**
(ii) Show that the quotient group

$$\mathsf{GL}(n, \mathbb{R})/\mathsf{SL}(n, \mathbb{R}).$$

is isomorphic to the multiplicative group $(\mathbb{R} \setminus \{0\}, \cdot)$, where $\cdot$ is the usual multiplication of real numbers. **8 marks.**

Problem 3: (16 marks)

For each $n \in \mathbb{N}$, let $U(n)$ be the group of units of $\mathbb{Z}_n$ under multiplication.

(i) Compute the multiplication Cayley table for $U(8)$, $U(10)$ and $U(12)$. Which of them are cyclic groups? Which of them are isomorphic between them? Justify your answer. **6 marks.**

(ii) Compute the order of $U(8)$, $U(10)$, $U(12)$ and $U(2024)$. **5 marks.**

(iii) Compute the inverse of $2023 \in U(2024)$ and the inverse of $5 \in U(2024)$. Justify your answer. **5 marks.**

Problem 4: (12 marks)

Let $(G, \cdot)$ be a group of order 6.

(i) Find all the possible order of all possible subgroups of $(G, \cdot)$. Justify your answer. **4 marks.**

(ii) Is there a subgroup of $(G, \cdot)$ with order 2? Is there a subgroup of $(G, \cdot)$ with order 3? Justify your answer. **4 marks.**

(iii) How many subgroups of order 3 can have $(G, \cdot)$? Justify your answer. **4 marks.**

Self-reflection

Describe your favourite result from the course and explain the reason. It is expected at least 10 lines to be counted. **5 marks.**