1. Suppose that G is a group and we ask the following question: exactly which groups can be homomorphic images of G.
Suppose that we know all normal subgroups of G. Using the homomorphism theorem (isomorphism theorem), explain in detail, how this information helps us to answer the above question.

[A group G' is a *homomorphic image* of G, if there exists a surjective homomorphism $G \to G'$.]

2. a) Suppose that G is a finite group, p a prime number and H a Sylow p-subgroup of the group G. Prove that H is the only Sylow p-subgroup of G, if and only if H is a normal subgroup of G.

b) Prove that every group of order 77 is cyclic.

3. a) Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 3 & 2 & 7 & 4 & 6 & 1 \end{pmatrix} \in S_7.$$

Present σ as a product of disjoint cycles. Present σ as a product of transpositions. Calculate $\operatorname{sgn}(\sigma)$.

b) Suppose that $\alpha, \beta \in S_n$ are any permutations. Prove that $\alpha\beta\alpha^{-1}\beta^{-1}$ is an even permutation.

4. a) What is the smallest natural number n, for which there exists a non-Abelian group of order n? Give an example of such a group.

b) Prove that a group G is Abelian, if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

5. Give three examples of non-isomorphic groups of order 27. Prove in detail that the groups indeed are non-isomorphic.

6. a) Suppose that G is a finite group, p a prime number and H a Sylow p-subgroup of the group G. Prove that H is the only Sylow p-subgroup of G, if and only if H is a normal subgroup of G.

b) Prove that no group of order 56 is a simple group.

7. Determine the composition factors of the group $D_{10} \times \mathbb{Z}_5$. Is the group solvable?

8. Prove that no group of order 200 is simple.

9. Prove that a subgroup of a solvable group is solvable.

10. a) Suppose that G is a group and $|G| = p^n q$, where p and q are prime numbers and $p > q$. Prove that the group G has exactly one subgroup of index q, and this subgroup is normal.

b) Prove by giving an example that the claim in item a) doesn't hold, if we replace the condition "$p > q$" by the condition "$p \neq q$".