Department of Mathematics and Statistics
Algebra II
2nd midterm exam 15.12.2025

1. Let

$$A = \{a + bi \mid a, b \in \mathbb{Z},\ a \equiv b \pmod 2\}.$$

Prove that A is a maximal ideal of the ring of Gaussian integers

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}.$$

2. Formulate and prove the universal property for the direct sum of modules:

$$\begin{tikzcd}
M_i \arrow[rr, "\varphi_i"] \arrow[dr, "\iota_i"'] & & N \\
& \bigoplus_i M_i \arrow[ur, "\theta"'] &
\end{tikzcd}$$

3. Which of the following polynomials are irreducible in the ring $\mathbb{Q}[X]$:
a) $3X^4 + 6X^3 + 3X^2 + 5X - 2$
b) $5X^4 + 18X^2 - 6X + 12$
c) $X^4 + aX^3 + bX^2 + cX + d$, where a, b, c, d are odd integers.

4. a) Prove that if K is a finite field, then $|K| = p^n$, where p is the characteristic of the field K and n is a positive integer. (4 p.)
b) Construct a field with 49 elements. (2 p.)