Department of Mathematics and Statistics
Algebra II
1st midterm exam 21.10.2024

1. a) Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 3 & 2 & 7 & 4 & 6 & 1 \end{pmatrix} \in S_7.$$

Present σ as a product of disjoint cycles. Present σ as a product of transpositions. Calculate $\operatorname{sgn}(\sigma)$.

b) Prove: If $H \leq S_n$ ($n \geq 2$), then either every $\sigma \in H$ is an even permutation or exactly half of the elements of H are even permutations.

2. Suppose that G is a group, in which $g^2 = e$ for every $g \in G$ (e is the neutral element of the group G). Prove that G is an Abelian group.

3. Prove that the group $(\mathbb{Q}, +)$ is not finitely generated (that is, there doesn't exist a finite subset $A \subset \mathbb{Q}$, for which $\langle A \rangle = \mathbb{Q}$). [Hint. You can prove first that $\langle m/n, m'/n' \rangle \subset \langle 1/nn' \rangle$.]

4. a) Suppose that G is a finite group, p a prime number and H a Sylow p-subgroup of the group G. Prove that H is the only Sylow p-subgroup of G, if and only if H is a normal subgroup of G.

b) Suppose that G is a group, $|G| = pn$, where p is a prime number and $p > n$. Also suppose that H is a subgroup of G and $|H| = p$. Prove that H is a normal subgroup of G.

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