Partial differential equations I — Final exam 2024, March 6.

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Answer to all four questions below. The exam lasts 120 minutes.

1. Give an example of a non-constant function $u = u(x, y, t)$, $u \in C^2(\mathbb{R}^2 \times \mathbb{R})$ that satisfies the wave equation in $\mathbb{R}^2 \times \mathbb{R}$.

2. Let $\psi \in C^2(\overline{\Omega})$ be a Neumann eigenfunction of the operator $-\Delta$ in a bounded smooth domain $\Omega \subset \mathbb{R}^n$ corresponding to the eigenvalue $\lambda \in \mathbb{R}$. This means that ψ is a non-zero function that satisfies

$$\begin{aligned} -\Delta\psi(x) &= \lambda\psi(x), \quad x \in \Omega, \\ \nu \cdot \nabla\psi|_{\partial\Omega} &= 0. \end{aligned}$$

   Prove that $\lambda \geq 0$.

3. Find a function $u(x, y, t) \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$ that is a solution of the transport equation

$$\begin{aligned} &\partial_t u(x, y, t) - 2\partial_x u(x, y, t) - \partial_y u(x, y, t) = 0, \quad (x, y) \in \mathbb{R}^2, \ t \in \mathbb{R}, \\ &u(x, y, t)|_{t=0} = \sin(x). \end{aligned}$$

4. Let $f : [0, \pi] \to \mathbb{R}$ be a C^2-function that satisfies $f(0) = f(\pi) = 0$ and $f'(0) = f'(\pi) = 0$. Let $u \in C^2([0, \pi] \times [0, \infty))$ be a solution of the heat equation

$$\begin{aligned} &\partial_t u(x, t) - \partial_x^2 u(x, t) = 0, \quad x \in [0, \pi], \ t > 0, \\ &u(x, t)|_{t=0} = f(x), \quad u(x, t)|_{x=0} = 0, \quad u(x, t)|_{x=\pi} = 0. \end{aligned}$$

   Show that

$$\lim_{t\to\infty} u(x, t) = 0$$

   for all $x \in [0, \pi]$.

   Hint: Consider the sine-Fourier series of f, that is,

$$f(x) = \sum_{k\geq 1} a_k \sin(kx), \quad a_k = \frac{2}{\pi}\int_0^\pi \sin(kx) f(x)\, dx.$$

Attached is material on Green's formulas which you are allowed to use in the exam.